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THE

PHYSICAL REVIEW.

IONIZATION AND EXCITATION OF RADIATION BY ELECTRON IMPACT IN NITROGEN.

By Bergen Davis and F. S. Goucher.

THE purpose of the present paper is to present some of the experimental results obtained in further experiments of the writers on ionization and excitation of radiation by electron impact in various gases.

The method of experimentation employed was in principle that used in connection with our experiments on mercury vapor and hydrogen and the reader is referred to our paper for a detailed description of the same.[1]

This method is simply a modification of that applied by Frank and Hertz and others for the determination of the ionizing potentials of various gases, with the end in view of differentiating between the effects due to radiation from the bombarded atoms and those due to ionization.

This modified method may be summarized by reference to the schematic diagram Fig. 1 as follows. Electrons from a source A are accel-

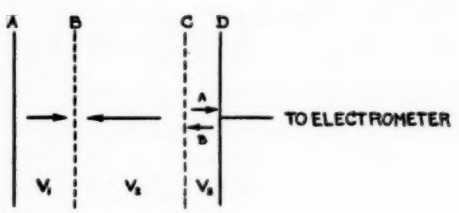


Fig. 1.

erated by an applied potential V_1 through a gauze B into a region BCD, C being a second gauze and D a collecting plate connected to an electrometer. V_2 is a retarding field applied between B and C, and V_2 is maintained at a constant amount greater than V_1 to prevent the original electrons from A reaching the plane of the gauze C. V_3 is small as compared with V_1 and V_2 and determines the direction of the field in the

[1] Physical Review, August, 1917.

neighborhood of C and D. The original electrons from A will be capable of making impacts with atoms or molecules in the space between A and C; and, depending on the energy imparted to them by V_1, may result in either the production of radiation from some of the impacted atoms or ionization or both. If the gauze C were not present and D in the place of C either of these results might cause a positive charging up of D due to the direction of V_2; if the cause were radiation alone it might cause a photoelectric emission of electrons from D, and if it were ionization alone the positive ions formed would travel to D. But with the small controlling field V_3 present the effects arising from these two causes can be differentiated, for if the cause be radiation alone it would produce a photo electric emission of electrons from either C or D dependent on the direction of V_3 and a consequent charging up of D—negatively if V_3 is in direction (a), and positively if in direction (b), if however the cause be ionization alone the positive ions formed in the region BC will be capable of reaching D in either case—V_3 being small as compared with V_2. As however both of these causes may occur simultaneously we may have a combination of these effects. Consequently the method consists in varying the applied potential V_1 maintaining V_2-V_1 at a constant small value, and obtaining two current curves as against V_1. One for V_3 being in direction (a); and the other for V_3 being in the direction (b). The shape of these curves and position of the intercept on the V_1 axis indicates the energy required to produce the different effects for any given gas.

The apparatus and experimental details were essentially the same as those used for mercury vapor and hydrogen. Certain improvements in design of the tube in which the measurements were made are worthy of note however. The same platinum thimble equipotential surface electron source was used, but the amount of metal in the tube was reduced considerably by using spiral wire grids wound on pyrex glass supports in place of the gauzes, with their heavy metal end supports. The grid B was of fine platinum wire about no. 30 and about 10 turns to the inch. The grid C was of nickel no. 20 and about 5 turns to the inch. The collecting electrode D was also of nickel and supported on a glass tube insulated by two grounding rings. All the seals were of glass, thus eliminating the use of cemented joints. This design made it possible to get rid of a much larger portion of the residual gases in the tube both on account of the reduction of the amount of metal and because the tube could be baked out as a whole and the grids glowed by means of current furnished from the outside.

The nitrogen used was prepared by the usual chemical method and

was purified by washing with H_2SO_4 and NaOH and then passing over hot copper. It was collected over distilled water and before being introduced into the measuring system was dried by passing through a phosphorus pentoxide tube.

The streaming method was employed; the pumps were kept running and the gas was allowed to pass through a porous plug inserted in the bore of a stopcock. By exposing more or less of the surface of the plug and regulating the speed of the pump desirable pressures could be maintained at a constant value as measured by a McLeod gauge.

Experimental Results.—From a large number of curves obtained under varying conditions of pressure and intensity of electron emission from the source, the curves shown in Figs. 2–5 were selected as being typical.

Fig. 2 shows that by reversing V_3 which was of the order of 2 volts in

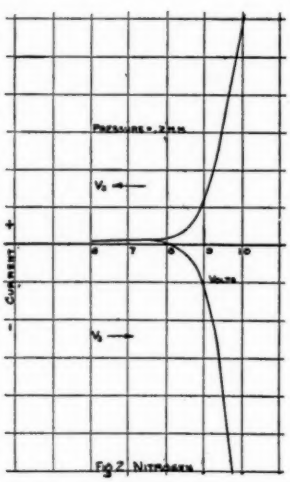


Fig. 2.

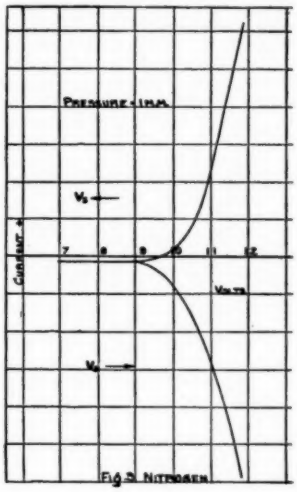


Fig. 3.

all these experiments the current curve could be made to reverse, indicating that the effect is produced largely if not entirely by radiation from the bombarded atoms. It is to be noticed that the curves have a horizontal portion in common. This is due to a very slight leak to the electrometer possibly from the glass. This could not seem to be avoided within small limits and was sometimes positive and sometimes negative. But the essential thing is that the two curves break away from this natural leak line at the same point which is about 7.5 volts, a result consistent with the results obtained by one of us and also as found by other experimenters.

There seems to be quite a marked increase in this radiation at about

9 volts; the curves being very steep after 9 volts. This is better shown in Fig. 3 where the electron emission was much less intense, here the curves do not separate until 9 volts had been reached. The current curve, Fig. 4, was obtained with a very small emission from the source and at low pressure a condition which is most favorable for ionization effects with this type of apparatus. This shows the effect to be radiation largely up to a value of about 17.5 to 18 volts. At this value the curve rises rapidly in spite of the opposing field V_3 showing that ionization has set in.

This then shows that the ionization potential of nitrogen is about 18 volts a value considerably higher than that heretofore supposed. This

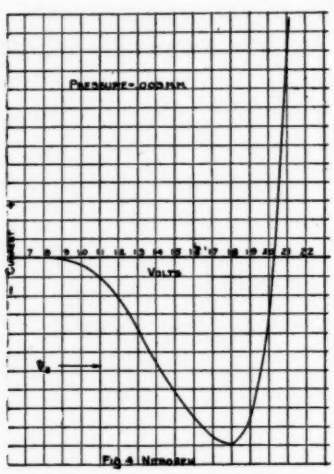


Fig. 4.

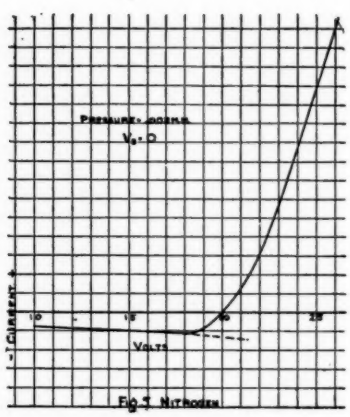


Fig. 5.

result however is more in accord with the work of Johnson on the total ionization by slow electrons. His straight line relation between total ionization and volts points more nearly to 20 than to 10 volts, though this method could not be expected to give an accurate value for the minimum ionization potential.

The intense radiation produced at about 9 volts is of interest in connection with the work of Meyer[1] who found that the wave-length about 1300 $A°$ stimulated the fluorescent band 3064 $A°$ in nitrogen. The value 1300 $A°$ from the quantum relation $Ve = hn$ would correspond to a voltage of the order of 9 applied to an electron.

Fig. 5 shows the current curve obtained when $V_3 = 0$. There is evidently not a complete neutralization of the radiation effect but the ionization effect is shown to set in strongly at about 18 volts.

[1] PHYSICAL REVIEW, November, 1917.

SUMMARY.

These experiments show.

(*a*) Radiation can be stimulated in nitrogen molecules by electron bombardment without any appreciable ionization of the same.

(*b*) The value 7.5 volts heretofore accepted as the value of the ionizing potential really corresponds to the least energy an electron must have in order that it may excite radiation by the bombarded molecule.

(*c*) There is an indication of a more intense type of radiation setting in at about 9 volts.

(*d*) That ionization sets in at about 18 volts.

These experiments were completed more than a year ago, but stress of circumstance has delayed their publication to the present time.

PHŒNIX PHYSICAL LABORATORY,
September, 1918.

ON A KINETIC THEORY OF MAGNETISM IN GENERAL.

BY KÔTARÔ HONDA AND JUNZÔ ÔKUBO.

1. In the last fifteen years, our knowledge of the nature of magnetism has made a great advance. The electron theory was first proposed by J. J. Thomson[1] and W. Voigt[2]; but their results were not conclusive. P. Langevin[3] developed a kinetic theory of paramagnetism as well as diamagnetism, and arrived at results which agree in many points with the observed facts; but there are many other facts which cannot be explained by his theory. Prof. P. Weiss founded his well-known theory of magnetons[4] on the basis of Langevin's theory of paramagnetic gas; but subsequent investigations[5] showed that the existence of the magneton —the fundamental unit of magnetism—is very doubtful. One of the present writers[6] modified Langevin's theory of the paramagnetism of gases so as to include the magnetization of liquid and solid states, and made clear the distinction between paramagnetic and diamagnetic substances. This theory explains the fact of the dependency of diamagnetic susceptibility on the temperature, and also its change with the change of state; while according to Langevin's theory, such changes of diamagnetism are improbable. Professor Weiss, by introducing a molecular field of enormously large magnitude (some 10^7 gausses) into Langevin's theory of paramagnetic gas, tried to explain the ferromagnetism; but although he succeeded in explaining some of the thermomagnetic properties of ferromagnetic substances, it is very difficult to understand from his theory, why magnetization is so easily induced by a small external field in the presence of the large molecular field. On several occasions, one of the present writers[7] has shown that the existence of the molecular field is very improbable and that the thermomagnetic properties accounted for by means of the molecular field can be equally well explained by other theories.

[1] Phil. Mag., 6, 1903, 673.

[2] Ann. der Phys., 9, 1902, 115.

[3] Ann. de chem. et phy., 5, 1905, 70.

[4] Arch. des Sci., 31, 1911, 401.

[5] K. Honda & T. Ishiwara, Sci. Rep., 4, 1915, 215. K. Honda & H. Takagi, Sci. Rep., 4, 1915, 261; Sci. Rep., 1, 1912, 229.

[6] K. Honda, Sci. Rep., 3, 1914, 171.

[7] K. Honda, Sci. Rep., 4, 1915, 208. K. Honda & J. Okubo, Sci. Rep., 5, 1916, 207. K. Honda, Sci. Rep., 7, 1918, 53.

Various theories of ferromagnetism have been advanced by Poisson, Weber, Maxwell and Ewing. Ewing's theory of molecular magnetism[1] explains the curve of magnetization and the phenomenon of hysteresis quite satisfactorily; but the qualitative nature of the theory is its great defect. R. Gans[2] tried to treat Ewing's model of molecular magnets mathematically; but his theory differs essentially from that of Ewing in assuming the distribution of the molecular magnets in the substance to be quite arbitrary, and in fact the conclusions from his theory are a rough approximation to the observed facts. In the last two years, we have published three papers,[3] in which Ewing's theory of molecular magnetism is treated quantitatively and extended so as to include the theory of hysteresis-loss, as well as the effect of temperature on magnetization. We have obtained the curve of magnetization possessing all the characteristics belonging to iron. Not only has the well-known hysteresis-loop been obtained, but the loss is expressed in terms of the magnetic field and the two constants characteristic of a substance The effect of temperature on the magnetization, that is, the complicated course of magnetization-temperature curves, is also explained on the basis of the same theory.

2. The theory of magnetism in general may thus be considered as established in the main; but there still remains a great advance to be made. As to Langevin's theories of paramagnetic and diamagnetic substances, some modifications are necessary in order to account for the observed facts. For example, take the case of diamagnetic substances: According to Langevin, the diamagnetic susceptibility k per unit of volume of a substance is given by

$$k = -\frac{\rho}{12}\left(\frac{e}{m}\right)^2 r^2,$$

where ρ is the density of electrons, e/m the ratio of the charge to its mass. and r the radius of the orbit of electrons. The susceptibility x per one gram atom or molecule is given by

$$x = -\frac{W\rho}{12D}\left(\frac{e}{m}\right)^2 r^2,$$

where W is the atomic or molecular weight of the substance and D its density. Now, for the density of electrons, we have

$$\rho = \frac{DN}{1800W},$$

[1] Phil. Mag., (5), 30, 1891, 205; Magnetic Induction in Iron and other Metals.

[2] Gött. Nachr., 1910, 197; 1911, 118. R. Gans & P. Hertz, Zeitsch. für Math. u. Phys. 61, 1913, 13.

[3] Sci. Rep., 5, 1916, 153; 5, 1916, 325; 6, 1917, 183.

provided each atom or molecule contains N electrons. Hence for different elements, we have

$$x = -\frac{N}{12 \times 1800}\left(\frac{e}{m}\right)^2 \Sigma r^2,$$

where Σ is to be extended for all electrons. Here we may take N as the atomic number of the elements. Thus, according to Langevin, since Σr^2 is to be considered as increasing with atomic weight, the diamagnetic susceptibility of the elements must increase with their atomic number. But, actually this is not the case; in fact, the susceptibility varies periodically with the atomic weight of the elements. This discrepancy is, however, partially reconciled, if we consider the diamagnetism actually observable to be the differential effect of the paramagnetic and diamagnetic phenomena;[1] that is, denoting the susceptibility observed by x, and Langevin's paramagnetic and diamagnetic susceptibilities by x_p and x_d respectively, we obtain

$$x = x_p + x_d;$$

hence though x_d increases proportionally with the atomic number, x may change periodically, provided x_p makes a similar change.

3. Next, consider the paramagnetic theory of Langevin. According to the kinetic theory of gases, the molecules of a gas are continuously making translational and rotational motions. We suppose each molecule to be a magnet composed of a set of revolving electrons. At a given instant, we may suppose the directions of the axes of these magnetic molecules to be uniformly distributed in all directions, if acted on by no external field. At the next instant, all the magnets change the directions of their axes; but as a whole, these axes are again equally distributed in all directions, and so on. In the case, when an external field acts on them, Langevin considers that the axes of these molecules tend to direct themselves toward the direction of the field; but the thermal impacts prevent this tendency. These two opposing effects make the distribution of the axes of the molecular magnets a little denser in the direction of the field than in the opposite direction. The directions of the magnetic axes are of course changing from instant to instant, but as a whole, the above distribution may be assumed to remain unchanged. By following the analogy of the distribution of the density of a gas under the action of gravity, he assumed the distribution of the axes of the molecular magnets to be given by Maxwell-Boltzmann's law. Resolving their magnetic moments in the direction of the field, and summing up these components, he obtained the well-known expression for the paramagnetic susceptibility.

[1] K. Honda, Sci. Rep., 3, 1914, 171; J. Kunz, PHYS. REV., 6, 1915, 113.

Now, since the molecules are continuously rotating, things are not so simple as Langevin considers. Generally the axis of rotation of a molecule does not coincide with its magnetic axis; hence we consider the magnetic moment to be decomposed into two components parallel and perpendicular to the axis of rotation. The former component is here termed the axial component of the moment and the latter the transverse component. When a magnetic field is suddenly applied to the molecule, in virtue of the axial component of the magnetic moment, its axis of rotation makes a small gyrostatic motion, that is, it makes procession and nutation round the direction of the field. This motion is very little affected by the presence of the transverse component of the moment. The amplitude of the nutational motion measured from the direction of the field extends from an initial angle α to an angle θ towards the field. This motion must therefore make a certain contribution to the increase of the magnetic moment in the direction of the field, while the precessional motion itself does not cause any change in the magnetic moment. If we suppose that by continuous thermal impacts with other molecules, the nutational motion, whose frequency is much larger than that of the precession, rapidly decreases, while the precessional velocity does not considerably alter, a simple precessional motion results with an angle θ, which is greater than the initial angle α and less than the angle θ, and causes also an increase of the magnetic moment in the direction of the field. If the field, instead of being applied suddenly, be gradually applied and increased extremely slowly to its final value, the molecules will make such slow nutational and precessional motions that the first precessional motion at the lower limit may be considered as statical. This case is analogous to that of a pendulum acted on extremely slowly by the force of gravity, in which case, the pendulum may be assumed to remain at rest at its lowest point. If, under the action of a magnetic field, the molecules make a simple precessional motion or take definite orientations, as explained above, it is evident that his case is almost the same as that treated by Langevin. Taking the sum of the magnetic moments in the direction of the field due to all molecules, we obtain usually a very small moment, that is, the paramagnetic moment. By increasing the field, the angular distance $\alpha-\theta$ increases for all molecules, and therefore the paramagnetism of the gas increases with the field. As will be shown later, this result almost coincides with that of Langevin.

4. Consider next the transverse component of the magnetic moment. This component of magnetism is continuously rotating with the molecule in its equatorial plane. Let us at first suppose that this plane coincides with the plane of the magnetic field. Initially its angular velocity is

uniform for all directions during a complete revolution; but if a magnetic field acts on the molecule, the revolution is now not uniform. Toward the direction of the field, the motion is always accelerated; but in the opposite direction, it is always retarded. Hence, if we wish to find its magnetic effect, it is necessary to take the time-mean of the magnetic moment during a complete period of revolution. Since during a revolution, the plus pole of the molecule is a shorter time in the direction of the field than in the opposite direction, the time-mean of the magnetic moment becomes negative, that is, the transverse component of the magnetic moment of rotating a molecule produces a diamagnetic effect. If the equatorial plane of the molecule makes some angle with the plane of the field, it is only necessary to resolve the above effect in the plane of the field. Hence, if there is a number of molecules, the directions of the equatorial planes of which are uniformly distributed between the angles 0 and π, we must take the component moments for these molecules in the direction of the field and integrate them; the result is a diamagnetism. Langevin did not in the least consider the effect corresponding to this kind of diamagnetism. The total magnetic effect of rotating molecules is then the differential effect due to axial and transverse components of magnetic moment, and therefore it may be paramagnetic or diamagnetic, according as the effect due to the axial component is greater or less than that due to the transverse component.

If the above view be correct, Langevin's theory of paramagnetic gas is not valid; and therefore the theory of magnetons, as well as that of molecular field put forward by Prof. Weiss, which are based on the Langevin theory, have no foundation. For this reason, the above view of the magnetization of a gas is very important.

5. We shall next consider the results qualitatively explained above, more in detail. Let us suppose that the magnetic field is acting on a

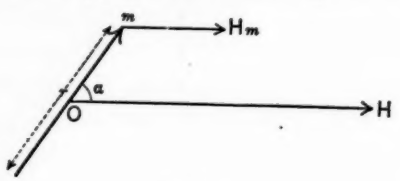


Fig. 1.

molecule of a gas rotating about its magnetic axis; then it will make precessional and nutational motions about the direction of the field. Let o be the center of the magnetic molecule, its pole-strength and pole-distance being m and r respectively. Let α be the initial angle of a molecular magnet, which is acted on by an external field H. Then the

magnetic moment M is

$$M = mr.$$

The following notations are here used:

K = moment of inertia about the magnetic axis,

ω = angular velocity about the magnetic axis,

$\dot{\psi}$ = precessional velocity,

$\dot{\theta}$ = nutational velocity.

From the theory of the gyroscope, we know that

(*i*) the angular momentum about oH remains constant,

(*ii*) the sum of the kinetic and potential energies of the gyro-molecule is constant;

that is,

$$K\omega \cos\theta + K\dot{\psi} \sin^2\theta = K\omega \cos\alpha,$$

$$\tfrac{1}{2}K\omega^2 + \tfrac{1}{2}K\dot{\psi} \sin^2\theta + \tfrac{1}{2}K\dot{\theta}^2 - MH \cos\theta = \tfrac{1}{2}K\omega^2 - MH \cos\alpha;$$

Hence

$$\dot{\psi} = \frac{K\omega(\cos\alpha - \cos\theta)}{K \sin^2\theta}$$

and

$$K\omega^2(\cos\theta - \cos\alpha)^2 + K^2\dot{\theta}^2 \sin^2\theta \, \dot{\theta} = 2MHK \, (\cos\theta - \cos\alpha).$$

The limits of the nutational angle θ and θ_1 are given by

$$\operatorname{Cos}\theta = \operatorname{Cos}\alpha,$$

$$\operatorname{Cos}\theta_1 = -\frac{1}{a} \pm \sqrt{1 + \frac{2}{a}\cos\alpha + \frac{1}{a^2}},$$

where

$$a = \frac{MH}{\tfrac{1}{2}K\omega^2}.$$

In the last equation, $\cos\theta$ must always be less than 1; hence the plus sign must be taken for the double sign. We have then

$$\theta = \alpha,$$

$$\cos\theta_1 = -\frac{1}{a} + \sqrt{1 + \frac{2}{a}\cos\alpha + \frac{1}{a^2}}.$$

Hence the gyro-molecule makes a nutational motion between α and θ_1.

Suppose at first that the magnetic field is applied infinitely slowly, or that the field is suddenly applied, but the nutational motion is assumed to subside very rapidly, the angle θ for the precessional motion being the same as in the above case. This last supposition may not be strictly true, but as we shall see later, if a be very small, the result does not materially differ from that of a case free from the above supposition. In a case, where a is small, the above supposition gives the result, which

exactly coincides with that of Langevin. The magnetic moment in the direction of the field is then $M\cos\theta_1$. The total magnetic moment per one gram molecule is

$$\sigma = \frac{Mn}{2}\int_0^{\pi} \cos\theta_1 \sin\alpha d\alpha$$

$$= \frac{Mn}{2}\int_0^{\pi}\left\{-\frac{1}{a}+\sqrt{1+\frac{2}{a}\cos\alpha+\frac{1}{a^2}}\right\}\sin\alpha d\alpha$$

where n is the number of molecules per one gram molecule of the gas. Writing $Mn = \sigma_0$, we have

$$\frac{\sigma}{\sigma_0} = \frac{1}{2}\int_0^{\pi_0}\left\{\sqrt{1+\frac{2}{a}\cos\alpha+\frac{1}{a^2}}-\frac{1}{a}\right\}\sin\alpha d\alpha$$

$$= \frac{1}{2}\int_{-1}^{+1}\sqrt{1+\frac{1}{a^2}+\frac{2}{a}x}\,dx - \frac{1}{a} = \frac{a}{6}\left|\sqrt{1+\frac{2}{a}x+\frac{1}{a^2}}\right|_{-1}^{+1} - \frac{1}{a}$$

$$= \frac{a}{6}\left|\left\{\left(1+\frac{1}{a}\right)^2\right\}^{\frac{3}{2}} - \left\{\left(1-\frac{1}{a}\right)^2\right\}^{\frac{3}{2}}\right| - \frac{1}{a}.$$

Hence if $a < 1$, we have, taking the minus sign for the square root of $\left(1-\frac{1}{a}\right)^2$,

$$\frac{\sigma}{\sigma_0} = \frac{a}{3}. \tag{1}$$

but if $a > 1$, we get, taking the plus sign for the square root,

$$\frac{\sigma}{\sigma_0} = 1 - \frac{1}{a} + \frac{1}{2a^2}. \tag{1'}$$

Since $\frac{1}{2}K\omega^2$ is the rotational kinetic energy of the molecule, a has the same meaning as that in the Langevin theory, that is,

$$a = \frac{MH}{rT},$$

where r is the gas constant referred to one molecule.

According to Langevin,

$$\frac{\sigma}{\sigma_0} = \operatorname{Coth} a - \frac{1}{a}; \tag{2}$$

if a be small, we have, by expansion,

$$\frac{\sigma}{\sigma_0} = \frac{1}{3}a - \frac{2}{90}a^3 + \cdots \tag{3}$$

which almost coincides with equation (1). In the following table, the two sets of values for σ/σ_0, as calculated from equations (1) and (2), are given side by side:

a	σ/σ_0 (Honda & Ôkubo).	σ/σ_0 (Langevin).	a	σ/σ_0 (Honda & Ôkubo).	σ/σ_0 (Langevin).
0.0	0.0000	0.0000	5.0	0.8133	0.8000
0.5	0.1667	0.1640	6.0	0.8426	0.8334
1.0	0.3333	0.3130	7.0	0.8638	0.8571
2.0	0.5833	0.5373	8.0	0.8802	0.8750
3.0	0.7037	0.6717	9.0	0.8930	0.8889
4.0	0.7708	0.7507	10.0	0.9033	0.9000

From the above table, we see that for small values of a, our formula exactly coincides with that of Langevin; but for a moderate value of a, these two slightly deviate from each other. In Fig. 2, these results (curves 1) are graphically given.

6. Next, we shall consider the case, when the magnetic field is applied

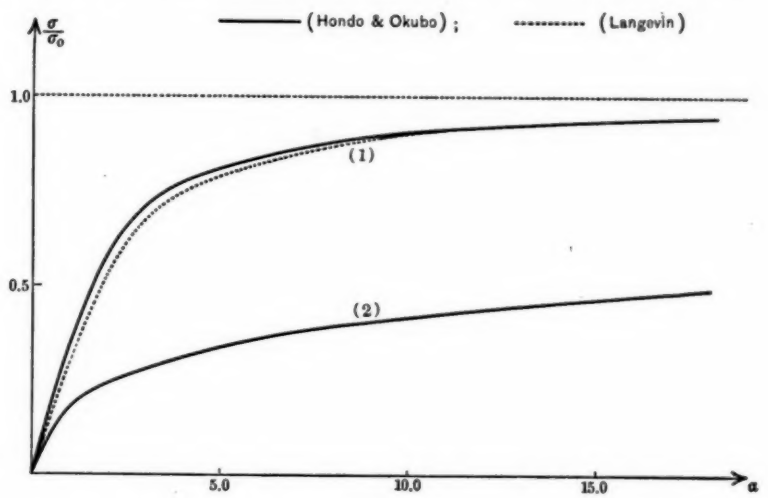


Fig. 2.

impulsively and the consequent nutational motion continues for a long time.

Now, the nutational velocity is given by

$$K \sin^2\theta \left(\frac{d\theta}{dt}\right)^2 = MH (\cos\theta - \cos\alpha)(\cos\theta - \cos\theta_1)(\cos\theta_2 - \cos\theta).$$

Putting

$$\cos\theta = \cos\theta_1 \cos^2\varphi + \cos\alpha \sin^2\varphi$$

and changing the variable from θ to φ, we get

$$\left(\frac{d\varphi}{dt}\right)^2 = \frac{MH}{4K}(\cos\theta_1 + \cos\theta_2)(1 - \kappa^2 \sin^2\varphi),$$

where

$$\kappa^2 = \frac{\cos\theta_1 - \cos\alpha}{\cos\theta_1 + \cos\theta_2}.$$

Hence

$$\frac{dt}{p} = \frac{d\varphi}{\sqrt{1 - \kappa^2 \sin^2\varphi}},$$

where

$$p = \sqrt{\frac{MH}{4K}(\cos\theta_1 + \cos\theta_2)}.$$

Let us suppose that

$$\text{at } t = T, \qquad \theta = \alpha, \qquad \varphi = \frac{\pi}{2},$$

and

$$\text{at } t = 0, \qquad \theta = \theta_1, \qquad \varphi = 0,$$

then

$$\frac{T}{p} = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - \kappa^2 \sin^2\varphi}} = K(\kappa),$$

where $K(\kappa)$ is the complete elliptic integral of the first kind.
θ or φ at any instant is given by

$$\frac{t}{p} = \int_0^{\phi} \frac{d\varphi}{\sqrt{1 - \kappa^2 \sin^2\varphi}} = F(\varphi, \kappa).$$

$F(\varphi, \kappa)$ is the Legendre elliptic integral of the first kind, that is,

$$\varphi = am\frac{t}{p}$$

and

$$\cos\theta = \cos\theta_1 \, cn^2\frac{t}{p} + \cos\alpha \, sn^2\frac{t}{p}.$$

Hence the time-mean of the magnetic moment is

$$\begin{aligned} M_m &= \frac{1}{T}\int_0^T M\cos\theta\, dt \\ &= \frac{M}{T}\left\{\cos\theta_1 \int_0^T cn^2\frac{t}{p}\,dt + \cos\alpha \int_0^T sn^2\frac{t}{p}\,dt\right\} \\ &= \frac{Mp}{T}\left|\cos\theta_1\left\{\frac{1}{\kappa^2}\left[E\left(am\frac{t}{p}, \kappa\right) - \kappa'^2\frac{t}{p}\right]\right\}\right. \\ &\quad \left. + \cos\alpha\left\{\frac{1}{\kappa^2}\left[\frac{t}{p} - E\left(am\frac{t}{p}, \kappa\right)\right]\right\}\right|_0^T. \end{aligned}$$

But since

$$\frac{p}{T} = K(\kappa) \text{ and } amK = \frac{\pi}{2},$$

$$\therefore \frac{M_m}{M} = \frac{\cos\theta_1 - \cos\alpha}{\kappa^2}\frac{E(\kappa)}{K(\kappa)} + \frac{1}{\kappa^2}(\cos\alpha - \kappa'^2\cos\theta_1)$$

$$= (\cos\theta_1 + \cos\theta_2)\frac{E(\kappa)}{K(\kappa)} - \cos\theta_2.$$

If the axis of the molecular magnets are initially distributed uniformly in all directions, the resultant magnetic moment in the direction of the field is

$$\frac{\sigma}{\sigma_0} = \frac{1}{2}\int_0^\pi \left\{(\cos\theta_1 + \cos\theta_2)\frac{E(\kappa)}{K(\kappa)} - \cos\theta_2\right\}\sin\alpha d\alpha. \qquad (4)$$

Since $\cos\theta_1$ and $\cos\theta_2$ are all functions of α, the above integral cannot be evaluated in a rational form. If a be small, $\cos\theta_1$, κ, $E(\kappa)$, $K(\kappa)$ can be expanded in the ascending power series of a and then integrated, we have

$$\frac{\sigma}{\sigma_0} = \frac{1}{3}a - 0.0334a^3 + \cdots. \qquad (5)$$

In a case when a is very large, these quantities can be expanded in an ascending power series of $\lambda = 1/a$; then we get

$$\frac{\sigma}{\sigma_0} = 0.4818 - 0.6875\lambda + 0.6476\lambda^2 + \cdots. \qquad (6)$$

This equation shows that even in an infinitely large field, the magnetic moment does not attain its saturation value; because for $H = \infty$, $\lambda = 0$. Hence

$$\frac{\sigma}{\sigma_0} = 0.4818.$$

The reason for this is evident, since according to our first supposition, all the molecules are making continuously nutational oscillations like a pendulum about the direction of the field.

By using equations (5) and (6) for small and large values of a, and also by mechanical integration of equation (4) for moderate values of a, we calculated the values of σ/σ_0 and the result is given in Fig. 2, curve 2. In ordinary cases actually realizable, in which a is a very small quantity, equations (3) and (5) coincide with each other; but as the values of a becomes large, the deviation between them becomes always greater. To decide which one of these equations for a large value of a actually corresponds to the observed facts, requires a further experimental investigation.

7. Next, we shall consider the case of a molecule rotating about an axis perpendicular to the magnetic axis. If the plane of rotation coincides with that of the field, the time-mean of the magnetic moment M_m in the direction of the field[1] is given by

$$\frac{M_m}{M} = \frac{2}{\kappa^2}\frac{E(\kappa)}{K(\kappa)} - \left(2\frac{\kappa'^2}{\kappa} + 1\right),$$

where

$$\kappa^2 = \frac{4MH}{K\omega^2} = 2a.$$

If the plane of rotation makes an angle θ with the direction of the field, its effective moment is $M_m \cos\theta$. Hence if the distribution of the planes of rotation for n molecules be uniform, the resultant moment in the direction of the field will be

$$\sigma = \frac{M_m n}{2}\int_0^\pi \cos\theta \sin\theta d\theta = \frac{M_m n}{2},$$

or

$$\frac{\sigma}{\sigma_0} = \frac{1}{2}\left\{\frac{2}{\kappa^2}\frac{E(\kappa)}{K(\kappa)} - \left(2\frac{\kappa'^2}{\kappa^2} + 1\right)\right\}. \tag{7}$$

If a or κ be small, we have by expansion

$$\frac{\sigma}{\sigma_0} = -\frac{a}{8} - \frac{a^2}{8} - \cdots. \tag{8}$$

This shows that the molecules rotating about the axis perpendicular to the magnetic axes produce the diamagnetism. Since $\frac{1}{2}K\omega^2$ in the expression for a is the rotational energy of the molecules, its value is of the same order of magnitude as rT. Hence even in the strongest field available at the present day, the value of a is very small, unless the temperature is very low. Hence we may safely neglect a^2 in comparison with a, and we obtain

$$\frac{\sigma}{\sigma_0} = -\frac{a}{8}. \tag{9}$$

8. Having thus found the magnetic effects of the gas molecules for the two cases above mentioned, we now proceed to consider the case of the magnetization of an actual gas. Since the molecules are always rotating about axes whose directions are continuously changing, the axis of rotation and the magnetic axis do not generally coincide with each other. Now, at any given instant, among n molecules, there is a certain number of molecules

$$dn = \frac{n}{2}\sin\varphi d\varphi,$$

[1] K. Honda & J. Okubo, Sci. Rep., 5, 1916, 328.

whose magnetic axis make with the axis of rotation an angle lying between φ and $\varphi + d\varphi$. Resolving the magnetic moment of these molecules in the axial and transverse directions, we get $M \cos \varphi$ and $M \sin \varphi$ for these two components respectively. On the one hand, these molecules have then the magnetic moment $M \cos \varphi$ in the direction of the axis of rotation, and therefore by applying a magnetizing field, their magnetic polarization is paramagnetic. Its intensity of magnetization $d\sigma$ is obtained from equation (1), since the quantity corresponding to a is very small; that is,

$$a' = \frac{MH \cos \varphi}{\frac{1}{2}K\omega^2} = a \cos \varphi$$

and

$$d\sigma = M \cos \varphi dn \times \frac{a'}{3} = \frac{Mna}{6} \cos^2 \varphi \sin \varphi d\varphi.$$

φ may take any value between 0 and π; hence integrating this expression for φ, we get the total effect, that is,

$$\frac{\sigma}{\sigma_0} = \frac{a}{6} \int_0^{\pi} \cos^2 \varphi \sin \varphi d\varphi = \frac{a}{9}. \tag{10}$$

On the other hand, the magnetic effect due to the transverse component of magnetic moment for the molecules is obtained from equation (9). Thus,

$$a' = \frac{HM \sin \varphi}{\frac{1}{2}K\omega^2} = a \sin \varphi$$

and

$$d\sigma = - M \sin \varphi dn \times \frac{a'}{8}$$

$$= - \frac{Mna}{16} \sin^3 \varphi d\varphi.$$

Integrating this expression for φ between 0 and π, we get the total effect, that is,

$$\frac{\sigma}{\sigma_0} = - \frac{a}{16} \int_0^{\pi} \sin^3 \varphi d\varphi = - \frac{a}{12}. \tag{11}$$

Hence, if the principal moments of inertia are equal to each other, that is, the shape of the molecules is spherical, the resultant moment of n molecules is the sum of the expressions (10) and (11). That is,

$$\frac{\sigma}{\sigma_0} = \frac{a}{3} (\tfrac{1}{3} - \tfrac{1}{4}) = 0.0278a. \tag{12}$$

Thus the resultant is paramagnetic. This differs quantitatively from the result of Langevin.

It is commonly assumed that the molecules of a monoatomic gas have a spherical shape and therefore thermal impacts cannot produce any rotation of the molecules and they have no energy of rotation. But according to a theory[1] proposed by one of the present writers, the molecules of the monoatomic gas are also continuously rotating with great velocity, which is acquired in a liquid state, but this velocity does not depend on the temperature of the gas. Hence, even for the monoatomic gas, the above result is equally applicable.

If the form of the molecules is a body of revolution about the magnetic axis, we have

$$a = \frac{MHK'}{\frac{1}{2}K^2\omega^2} = \frac{\sigma_0 H}{q}\frac{K'}{K}$$

and

$$a' = \frac{MH}{\frac{1}{2}K'\omega'^2} = \frac{\sigma_0 H}{q},$$

where K and K' are respectively the moments of inertia about the magnetic axis and the axis perpendicular to it, and

$$q = \frac{n}{2}K\omega^2 = \frac{n}{2}K'\omega'^2$$

is the kinetic energy of rotation, the equipartition of the rotational energy being assumed. Hence,

$$\frac{\sigma}{\sigma_0} = \left(\frac{K'}{3K} - \frac{1}{4}\right)\frac{\sigma_0 H}{3q}. \tag{13}$$

The gas may therefore be paramagnetic or diamagnetic, according as

$$4K' - 3K \gtrless 0.$$

As an example, take an ellipsoidal molecule, whose form is defined by

$$\frac{x^2}{A^2} + \frac{y^2 + z^2}{B^2} = 1$$

and whose major axis x coincides with that of its magnetic axis; then we have

$$K = \frac{M}{5} \times 2B^2 \text{ and } K' = \frac{M}{5}(A^2 + B^2).$$

Hence the gas is paramagnetic or diamagnetic according as

$$4A^2 - 2B^2 \gtrless 0,$$

or

$$\frac{A}{B} \gtrless 0.7072,$$

or

$$\text{eccentricity} \gtrless 0.707.$$

[1] K. Honda, Sci. Rep., 7, 1918.

It is an interesting fact that the sign of the magnetization of a gas depends only on the shape of the molecules, and not in the least on their the magnetic moment.

9. It is generally admitted that the molecular magnets consist of certain sets of electrons rapidly revolving about positive nuclei, and that this revolution of electrons does not vary with temperature, its velocity always remaining the same, even at absolute zero. This rotational energy may be called the zero point energy; we denote it by $\frac{1}{2}R\Omega$, where R is the gas constant referred to one gram molecule. Besides, at the temperature T, we have also the axial component of rotation of the molecules as a whole; its energy may be given by $\frac{1}{2}RT$. We have then

$$\frac{n}{2}K\omega^2 = \tfrac{1}{2}RT + \tfrac{1}{2}R\Omega = \tfrac{1}{2}R(T+\Omega).$$

Hence equation (13) becomes

$$\frac{\sigma}{\sigma_0} = p\frac{\sigma_0 H}{R(T+\Omega)},$$

where

$$p = \frac{2}{3}\left(\frac{K'}{3K} - \frac{1}{4}\right);$$

p may be positive or negative; or

$$x = \frac{\sigma}{H} = \frac{p\sigma_0^2}{R(T+\Omega)}. \tag{14}$$

Thus, the paramagnetic or diamagnetic susceptibility is proportional to the magnetic field. From the last equation, we have

$$x(T+\Omega) = \frac{p\sigma_0^2}{R} = \text{const.}$$

If $p>0$ and Ω is very small as compared with T, we have

$$xT = \text{const.},$$

which is the Curie law for paramagnetic gas. On the other hand, if $p<0$ and Ω is very large, we have

$$x = \text{const.},$$

Hence for a gas satisfying these conditions, its diamagnetic susceptibility is independent of temperature.

Here the following remarks are to be made. So far we have assumed the molecular magnets to be perfectly rigid; but as shown by Langevin, the motion of the electrons is slightly modified by applying a strong magnetic field, and this change also contributes to the diamagnetism of

the substance. Hence to equation (14), we must add the diamagnetic susceptibility as given by Langevin, that is,

$$x = -\frac{N}{12 \times 1800}\left(\frac{e}{m}\right)^2 \Sigma r^2.$$

As an application of the above theory, it is interesting to calculate the magnetic moment of a hydrogen atom or molecule. According to Bohr, the atom of hydrogen consists of an electron revolving about its nucleus. Since Bohr's value for the smallest radial distance of the revolving electron seems to be too small, we take its next value for r and the corresponding angular velocity, that is,

$$r = 2.2 \times 10^{-8} \text{ cm.}, \qquad \omega = 0.755 \times 10^{15} \text{ per sec.}$$

Then the magnetic moment of an atom is

$$M = \tfrac{1}{2}er^2\omega = 2.95 \times 10^{-21}$$

and

$$\sigma_0 = 1806,$$

which is a little greater than half the magnetic moment of the nickel atom.

The zero point energy E_0 of rotation amounts to

$$E_0 = 6.2 \times 10^{11};$$

but at 27° C.,

$$\tfrac{1}{2}RT = 1.24 \times 10^{10},$$

which is only a small fraction of the zero point energy. Hence, we have

$$\frac{n}{2}K\omega^2 = E_0 + \tfrac{1}{2}RT = 0.63 \times 10^{11}.$$

In the well known Bohr model of a hydrogen molecule, the two electrons are revolving about the line joining two positive nuclei. Here the moment of inertia K about the magnetic axis is very small as compared with the moment of inertia K' about an axis perpendicular to it; hence the ratio K'/K is very large and therefore the susceptibility of hydrogen must be paramagnetic. This inference, however, contradicts the observed fact. In order therefore that the hydrogen molecule may give the observed diamagnetic susceptibility, the positive nuclei must approach very near to each other.

Let us next calculate the distance d between two positive nuclei of a hydrogen molecule composed of two atoms, the planes of the electron orbits being parallel to each other, under the condition that the susceptibility of the gas may give the value[1] actually observed, that is,

$$x = -1.89 \times 10^{-6}.$$

[1] The measurement of the susceptibility of different gases is now being made in this laboratory by Mr. J. Soné.

In this case, Langevin's diamagnetic susceptibility amounts to

$$x = -7.0 \times 10^{-6}.$$

Hence by equation (13), we have the relation

$$-1.8 \times 10^{-6} = \frac{1}{3}\left(\frac{K'}{3K} - \frac{1}{4}\right)\frac{\sigma_0^2}{q} + xd;$$

or

$$\frac{K'}{3K} = \frac{1}{12} + 5.4 \times 10^{-6}\frac{q}{\sigma_0^2} = 1.17.$$

$$\therefore \quad \frac{r}{d} = 13.12.$$

Hence the two positive nuclei should be only distant from each other by an amount less than one thirteenth of the radius of the electron orbits.

10. The case of solid substances will be next considered. According to the prevailing theory of the solid state, the molecules are arranged according to a certain space-lattice and are continuously making rectilinear oscillations, but not any rotations. But as we have already remarked, the electrons in each atom are rapidly revolving about the centers of mass of the electrons and the positive nuclei; its axis of rotation coincides evidently with that of the magnetic moment. The energy of this rotational motion, that is, the zero point energy, may be expressed by $\frac{1}{2}\epsilon R\Omega$, where Ω is a constant and ϵ a small fraction of unity. With the rise of temperature, the angular velocity of this rotation may probably increase, because of the axial rotation of the molecules as a whole due to thermal impacts. In a solid, this increase of rotation is not independent of the rectilinear vibrations and therefore the increase of energy is far less than that corresponding to the energy of one independent freedom, that is, than $\frac{1}{2}RT$; hence it may be expressed by $\frac{1}{2}\epsilon RT$. For the molecules of a solid, a simultaneous rotation of the molecules perpendicular to the above rotation can not possibly exist.[1] Hence assuming the molecules to form a numerous number of minute groups, in each of which the axis of magnetic molecules take the same direction, but the magnetic axes of these groups are uniformly distributed in all directions, the expression for the susceptibility of a solid substance is given by equation (1). Since

$$\frac{n}{2}K\omega^2 = \frac{\epsilon}{2}R(T + \Omega),$$

$$\frac{\sigma}{\sigma_0} = \frac{a}{3} = \frac{2}{3}\frac{\sigma_0 K'H}{KR\epsilon(T + \Omega)};$$

[1] K. Honda, Sci. Rep., 7, 1918.

$$x = \frac{\sigma}{H} = \frac{2}{3}\frac{\sigma_0^2 K'}{KR\epsilon(T+\Omega)}; \qquad (15)$$

$$\therefore \quad x(T+\Omega) = \frac{2}{3}\frac{\sigma_0^2 K'}{KR\epsilon} = \text{const.}$$

There is a considerable number of paramagnetic substances, whose susceptibility satisfies the above relation. Strictly speaking, to the above value of susceptibility, we must also add the small diamagnetic susceptibility given by Langevin.

According to the theory of fusion[1] proposed by one of the present writers, the rotational motion of the molecules during fusion begins to take place with the energy corresponding to the degree of freedom at the melting point, so that the liquid molecules are making axial and transverse rotations like those of a gas. Hence during fusion, the susceptibility of a substance discontinuously changes from a value expressed by equation (15) to that expressed by equation (14), and may in some cases change its sign. The remarkable fact[2] that during melting, the susceptibility of tin changes from paramagnetic susceptibility to the diamagnetic, is explained in this way.

11. The case of a ferromagnetic substance comes next for our consideration. According to our theory of magnetism,[3] which explains the observed fact in a very satisfactory manner, the molecules of a ferromagnetic substance are arranged in a definite space-lattice and exert mutual magnetic action upon one another. In the light of the present theory, this means that the energy of rotation about the magnetic axis, that is $\frac{1}{2}K\omega^2$, is very small; for in this case, the gyrostatic action of the molecules is negligably small in comparison with the mutual magnetic action, and the molecules are now in equilibrium under the action of this magnetic force, a condition required in our theory.

Now, take equations (1) and (1′) as our starting point; it appears at first sight that the expression

$$\frac{\sigma}{\sigma_0} = \frac{a}{3} \text{ for } a < 1,$$

$$\frac{\sigma}{\sigma_0} = 1 - \frac{1}{a} + \frac{1}{2a^2} \text{ for } a > 1,$$

where

$$a = \frac{\sigma_0^2 HK'}{\frac{n}{2}K^2\omega^2},$$

[1] K. Honda, Sci. Rep., 7, l. c.

[2] K. Honda, Sci. Rep., 1, 1911, 1.

[3] K. Honda and J. Okubo, Sci. Rep., 5, *l. c.*

may be looked on as the law of magnetization for a ferromagnetic substance, provided we consider σ to be a function of H and $(n/2)K\omega^2$ to be a very small quantity depending on the temperature of the substance. The form of the $\sigma_1 H$ curve is the same as curve 1 in Fig. 2. The curve possesses two characteristic points of the curve of magnetization, that is, the proportionality between the initial magnetization and the field, and the gradual approach of magnetization to saturation; yet it is only a rough representation of the fact; moreover it cannot explain the important phenomenon of hysteresis. These discrepancies are the consequences of the present theory, which neglects the mutual magnetic action and takes only the axial rotation of the molecules into account. As we have shown in our former papers, in the case of ferromagnetic substances, we must consider the mutual magnetic action as the most important factor in determining the law of magnetization, and therefore equations (1) and (1′), which take only the gyrostatic action of the axially rotating molecules into account, cannot be considered as the true law of magnetization.

The effect of temperature on magnetization will be next considered. The temperature affects magnetization in two opposite ways; the first effect, which is especially conspicuous in weak fields, but which completely vanishes in strong fields, is to increase the magnetization, while the second is always to decrease the magnetization. In our former paper,[1] these two effects are explained on the basis of small vibratory motions of the molecular magnets, about their mean orientations. That the magnetization in weak fields is considerably increased by the thermal vibrations of the molecules, is very satisfactorily explained; but the explanation of the second effect by the same vibrations is met with the great difficulty that if the half-amplitude of this vibration exceeds about 131°, the magnetization in a strong field is reversed; this, however, is an abnormal result, which has never been observed.

In view of the present theory, the following explanation for the temperature-effect in question will be very natural and agrees satisfactorily with the observed facts: The molecules of a ferromagnetic substance are continually making axial rotations about their magnetic axis, and at the same time, small vibratory motions about the axes perpendicular to it. The first motion explains the diminution of magnetization, while the second motion explains its increase in weak fields, as shown in our former paper. At ordinary temperature, the energy of the axial rotation is considered to be very small, but as the temperature rises, it rapidly increases, its rate of increase becoming always greater. At the critical

[1] Sci. Rep., 5, 1916, 325.

point, the energy is considered to be greatly increased, but it has still a small fraction of the value $\frac{1}{2}RT$ corresponding to one degree of freedom at the temperature T. The energy-content of the axial rotation is then given by an expression of the form

$$\frac{n}{2}K\omega^2 = \frac{m}{2}RT,$$

where m is a function of temperature increasing rapidly with the latter, the zero point energy being neglected. An expression m of the form

$$m = \frac{\epsilon}{\sqrt{1 - \frac{T}{\theta} + \epsilon}},$$

where θ is the critical temperature; e and e_1 being very small quantities, will accord with the observed facts very satisfactorily. The law of variation of magnetization with temperature is then given by equations (1), (1′) and

$$a = \frac{\sigma_0^2 HK'}{\frac{1}{2}KmRT}.$$

In order to show that these expressions are a good representation of the

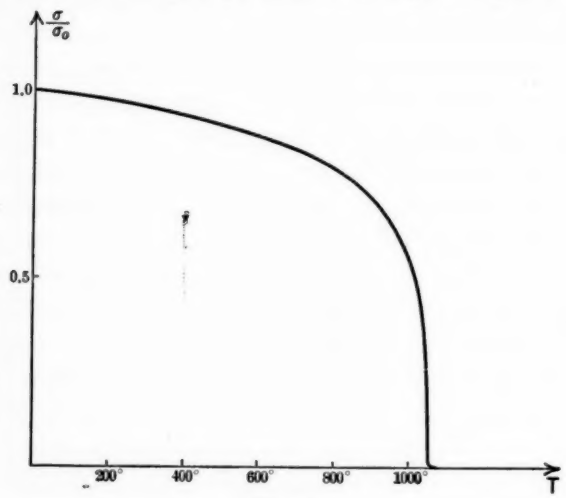


Fig. 3.

observed facts, we have calculated the constants entering into the expression of a for iron and obtained the following results:

$$\theta = 790° \text{ C.},$$

$$\epsilon = 0.431 \times 10^{-5},$$

$$\epsilon_1 = 4.96 \times 10^{-8}.$$

The curve is also given in Fig. 3; we see that it is very similar to the I_1T curve in a strong field. Hence we conclude that the effect of temperature, which diminishes the magnetization for all fields, is due to the increase of the axial rotational energy of the molecules. This inference is also confirmed by the fact that during heating or cooling through the critical range of magnetic transformation, the heat is absorbed or evolved, that is, through the same range, the specific heat abnormally changes.

If the above view be correct, the rotational energy gained during heating through the range of magnetic transformation up to the critical point, is equal to the total heat absorbed during the transformation. If we neglect a small rotational energy at room temperature, the total energy q gained is equal to $(n/2)K\omega^2$. Hence, for one gram-molecule of a ferromagnetic substance, we have

$$x = \frac{2}{3}\frac{K'\sigma_0^2}{nK^2\omega^2} \text{ or } = \frac{K'\sigma_0^2}{3Kq};$$

$$q = \frac{K'\sigma_0^2}{3KX}.$$

This relation admits of the calculation of the heat absorbed or evolved during heating or cooling, provided the shape of the molecules be known. But at present, we have no data for determining the ratio K'/K for ferromagnetic substances, and therefore the verification of the above relation is left for a future occasion.

The following remark deserves also to be made:—The magnetic (or A_2) transformation of iron has been the subject of a long dispute among metallurgists, viz. as to the nature of this transformation.[1] The present theory gives, however, an exact mechanism of this transformation; that is, the A_2 transformation is not a change of molecular configuration or a change in the molecules, as is the case in an allotropic change; but it is the process of acquiring the rotational energy of the molecules, whose amount is a definite function of temperature. Hence the transformation cannot be considered as an allotropic change, as has repeatedly been stated by one of the present writers.[1]

10. Lastly let us consider the susceptibility of a ferromagnetic substance above its critical point. In this region, the kinetic energy of axial rotation, that is, $(n/2)K\omega^2$, is very large in comparison with that in the magnetic region, but it is still far less than the energy corresponding to one degree of freedom at the temperature under consideration, that is, $\frac{1}{2}RT$, which value is only attained by the fusion of the substance. We may express the above fact by the expression

$$\tfrac{1}{2}K\omega^2 = \tfrac{1}{2}RT - \tfrac{1}{2}R\Omega$$

[1] K. Honda, Sci. Rep., 4, 1915, 169; 6, 1917, 213.

and therefore equation (1) becomes

$$\frac{\sigma}{\sigma_0} = \tfrac{1}{3}a = \frac{2\sigma_0 HK'}{KR(T-\Omega)},$$

or

$$\varkappa(T-\Omega) = \frac{2\sigma_0^2 K'}{RK} = \text{const.} \qquad (16)$$

This is the relation, which holds good approximately above the critical point. This relation was first obtained by Prof. P. Weiss from his theory of molecular field and given in his paper as a confirmation of his theory. But as was shown by one of the present writers, it can also be obtained from another theory,[1] which has no connection with the molecular field. The above deduction of the same relation affords also a similar example.

The relation (16) may also be written as

$$I(T-\Omega) = \text{const.} \times H.$$

This represents a hyperbola, whose parameter is H; curves 1, 2, 3, in Fig. 4 are those corresponding to successively increasing fields. Curie,

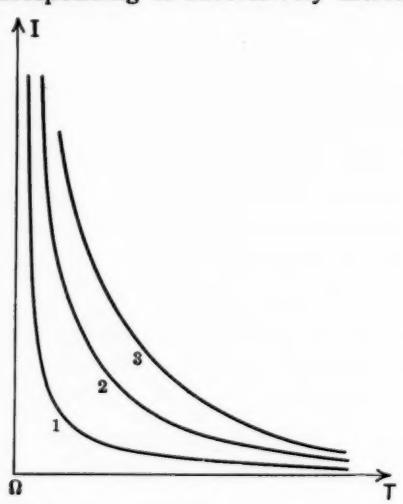


Fig. 4.

in his experimental research on the susceptibility of iron at high temperatures, first noticed this result and remarked that the course of the curves resembles the p, v curve of a gas near its critical point. In recent years, this remark has attracted the attention of many physicists[1]; they have attempted to explain the relation between ferromagnetic and paramagnetic states by the analogy of the relation of the liquid state to the gaseous. But as we have shown above, the fact is a natural consequence from the present theory, and we think it has not the important significance which other physicists assign to it.

[1] P. Weiss & P. N. Beck, Jour. de Phys., 7, 1908, 249. Ashworth, Phil. Mag., 27, 1915, 357; 30, 1915, 711; 33, 1917, 334. E. Bruins, Phil. Mag., 34, 1917, 380.

ON THE THEORY OF SUPERPOSED DIFFRACTION-FRINGES.

By Chandi Prasad.

IN the Proceedings of the American Philosophical Society for 1913 (pp. 276–282) C. F. Brush[1] has published various interesting observations of diffraction phenomena which he has attempted to group together under a common explanation by regarding them as "superposed fringes." Among the effects observed by him were those due to the diffraction of light by a cylindrical edge which have recently been further investigated and explained in an entirely different manner by N. Basu.[2] In the present paper, I am concerned with what were perhaps the most interesting and original observations recorded by Brush, that is, the diffraction-effects produced by a row of straight edges placed in echelon order. The arrangement adopted by Brush is shown diagrammatically in Fig. 1. A row of Gillete razor-blades (sometimes as many as twenty-four) were clamped together, so that their edges lay as nearly as possible in one plane which was placed very obliquely in the path of the train of light waves. The phenomena in the immediate neighborhood of the system were observed through a microscope. Brush noticed that the fringes showed contrasts between the maxima and minima of illumination which were much more marked than in the diffraction fringes of the Fresnel type due to a single straight edge, and explained this as due to the superposition of the fringe-systems produced by the successive edges. I propose in the present paper to show how the principle of superposition suggested by Brush may be formulated mathematically and its validity tested in experiment. The value of the principle is that it simplifies the treatment of the problem of diffraction by a succession of edges which

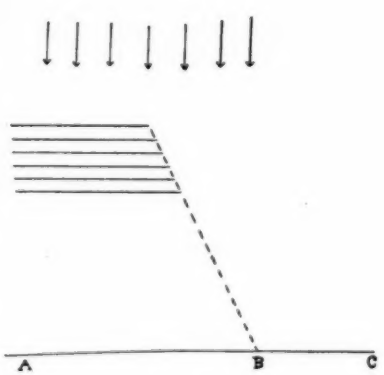


Fig. 1.

[1] See also Science Abstracts (1913), No. 1810.

[2] Phil. Mag., Jan., 1918, p. 79.

would otherwise be very laborious.[1] Incidentally I also describe some observations of my own which appear to be of interest from the point of view of the general theory of diffraction.

Sommerfeld has shown in his well-known investigation on the mathematical theory of diffraction[2] that the diffraction fringes due to a semi-infinite screen may be regarded as due to the interference of a system of a series of cylindrical waves emitted by the edge of the screen with the incident plane waves. In the case of a number of semi-infinite screens placed parallel to each other in echelon order, it would obviously *not* be correct to assume that in the region to the rear of the system, each of the edges emits cylindrical waves, the amplitude of which, in any direction is the same as if the other screens were absent. The assumption would certainly be incorrect in respect of the region lying on one side of the plane containing the edges (*AB* in Fig. 1); for, in this region each of the screens would obviously intercept and cut off the radiations emitted by the edges in front of it. But in the region lying on the other side (*BC* in Fig. 1), an assumption of the kind stated above would be justified as a first approximation, provided each edge in the echelon is sufficiently in advance of the preceding edge in relation to its distance from it. For, the illumination in the region at and near each edge would then be practically the same as that due to the incident waves alone, and the edge of each screen therefore emits a radiation, the amplitude of which in the region considered is practically the same as in the absence of the other screens. By superposing the effects of the cylindrical waves emitted by the edges upon that due to the incident waves, a mathematical treatment of the phenomena observed by Brush would be possible.

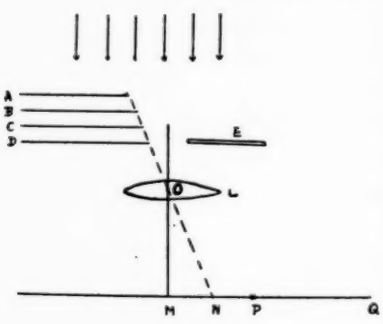


Fig. 2.

In order to test the preceding views and to obtain a definite confirmation, I have devised the experiment shown diagrammatically in Fig. 2 in which the effects due to the echelon of edges may be examined separately from that due to the incident waves.

A row of razor edges is placed on an optical bench, the distances between successive edges being about 10 cm. The edges are brought carefully into parallelism with each other and placed in echelon order.

[1] On this point, see Mascarat's Traite de Optique, Vol. I., p. 287.

[2] Math. Annalen, Vol. XLVII., 1896, p. 317.

The incident plane waves of light are diffracted by the edges and pass through the lens L, the observations being made in the focal plane. The superfluous light was cut off by the screen E. The lens may be dispensed with, if the fringes are examined at a great distance behind the system. Some interesting effects are then observed. First, we have the direct light which comes to a focus at or near the point M, on either side of which we have the diffracted light due to the limitation of the aperture of the lens. In the region NPQ, we have another system of fringes which are due to the superposition of the light diffracted by the series of edges. These fringes appear only to the right of the plane containing the edges, ON being the line drawn through the center of the lens parallel to this plane. The center of this system of fringes (as seen in white light) is at P where $MN = NP$. Fig. 3A and Fig. 3B reproduce photographs of the system of fringes observed with an echelon of two edges only, the second edge being further in advance of the first edge in the latter case than in the former. It will be seen that the fringe-widths are asymmetrical, that is, the fringe width decreases continuously as we move from one side of the system to the other. The positions of the fringes are approximately the same as that due to the interference of two cylindrical sources of light placed respectively at the two diffracting edges. This is shown by the following measurements of the width of the fringes on either side of the center P.

	Positions of the Maxima Read on a Micrometer Scale.	Fringe Widths Observed.	Fringe Widths Calculated.
First Maxima.....	3.283	.125	.124
Second Maxima...	3.158	.105	.098
Third Maxima....	3.053	.094	.084
Fourth Maxima...	2.959	.071	.075
Fifth Maxima.....	2.888	.069	.068
Sixth Maxima.....	2.819 (the central fringe)	.062	.063
Seventh Maxima..	2.757	.056	.058
Eighth Maxima...	2.701	.051	.054
Ninth Maxima....	2.650	.051	.052
Tenth Maxima....	2.599	.045	.052
Eleventh Maxima.	2.554	.045	.052
Twelfth Maxima..	2.509		

The figures in the last column were calculated from the formula

$$a(\cos\theta - \cos\psi) = \pm n\lambda,$$

where a is the distance between the two edges; θ, ψ, are the small angles made by the incident and diffracted rays, respectively, with the plane containing the edges.

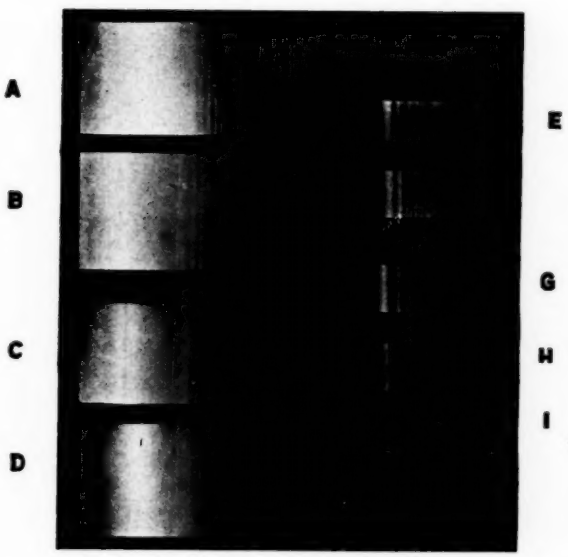
Fig. 3*C* reproduces a photograph of the fringes obtained with an echelon of three edges and Fig. 3*D* reproduces one obtained with four edges. It will be seen that in the former case, we have one faint secondary maxima between each pair of primary maxima, and in the latter case, we have two secondary maxima between the principal maxima. In fact, the results are somewhat though not completely analogous to those due to a diffraction grating composed of a small number of elements. The difference in this case is that the light is diffracted by simple edges, while in an ordinary diffraction grating we are concerned with either a corrugated surface or the strips of a plane surface. Photographs of the diffraction spectra formed by a very obliquely held surface consisting of two or three reflecting strips in a plane have recently been published by Mitra,[1] who has observed that the corresponding bands on either side of the pattern are of unequal brightness, the wider fringes on one side being of very feeble intensity in comparison with the narrow fringes on the other. The photographs of the diffraction effects produced by a row of straight edges published in the present paper (Figs. 3*A* to *D*) do not show this effect, thus indicating an essential difference between the two cases. In fact, in the diffraction spectra due to a row of straight edges, the broader fringes on one side are, if anything, actually brighter than the narrow fringes on the other.

With a view to further study of the effects noticed by Brush (in which the radiations from the edges are superposed on the transmitted waves), I have also measured and photographed the diffraction-fringes of the Fresnel type due to two edges lying in nearly the same line as the direction of propagation of the incident light. Four photographs are reproduced in Figs. 3 (*E* to *H*), the second edge being in advance of the first edge by different distances in the four pictures. The diffraction-fringes due to a single edge are shown in Fig. 3*I* for comparison. Some of the minima of illumination due to the superposition of the effects of the two edges are seen in the photographs to be much darker than the minima in the fringes of the Fresnel type due to a single edge. If we assume as a first approximation that the effects due to the two edges are practically additive, the illumination at any point in the fringe system may be calculated mathematically as follows:

The problem of the diffraction of light by a semi-infinite screen has been solved by Sommerfeld whose solution of the equation

$$\frac{\partial^2 s}{\partial t^2} = a^2\left(\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2}\right)$$

[1] Philosophical Magazine, Jan., 1918, Plate V.



SUPERPOSED DIFFRACTION FRINGES.

FIG. 3.

C. PRASAD.

is

$$s = \frac{e^{i(nt+\pi/4)}}{\sqrt{\pi}}\left[e^{-ikr\cos(\phi-\phi')}\int_{-\infty}^{\sqrt{2kr}\cos(\phi-\phi')/2} e^{-iT^2}dT \pm e^{-ikr\cos(\phi+\phi')}\int_{-\infty}^{\sqrt{2kr}\cos(\phi+\phi')/2} e^{-iT^2}dT\right].$$

This solution may be simplified by expanding the integrals in a semi-convergent series, as shown by Sommerfeld. In the region of light where $\pi - \varphi' < \varphi < \pi + \varphi'$, and provided $\sqrt{2kr}\cos(\varphi - \varphi')/2 > 1$, we have $s = \cos(kr\cos\varphi - \varphi' + nt)$

$$+\frac{1}{4\pi}\sqrt{\frac{\lambda}{r}}\cos\left(kr - nt + \frac{\pi}{4}\right)\left\{\frac{\pm}{\cos\frac{\varphi+\varphi'}{2}} - \frac{1}{\cos\frac{\varphi-\varphi'}{2}}\right\}.$$

In the case of normal incidence, $\varphi' = \pi/2$, and writing $\varphi' = 3\pi/2 - \delta$, where δ is small, we find on reduction and simplification

$$s = \cos(kd - nt) + \frac{\sqrt{\lambda d}}{2\pi x}\cos\left(kd - nt + \frac{kx^2}{2d} + \frac{\pi}{4}\right),$$

which gives the usual maxima and minima when

$$\frac{kx^2}{2d} + \frac{\pi}{4} = 2n\pi \quad \text{or} \quad (2n+1)\pi,$$

respectively. Now, if another edge were placed at a distance a cm. and projecting b cm. from the line of the geometrical shadow of the first edge, and its effect is added to that of the latter,

$$s = \cos(kd - nt) - \frac{\sqrt{\lambda d}}{2\pi x}\cos\left(kd - nt + \frac{kx^2}{2d} + \frac{\pi}{4}\right) - \frac{\sqrt{\lambda(d-a)}}{2\pi(x-b)}\cos\left(kd - nt + \frac{k\overline{x-b}^2}{2(d-a)} + \frac{\pi}{4}\right).$$

This gives for the intensity of illumination at any point

$$J = 1 + \frac{\lambda d}{4\pi^2x^2} + \frac{\lambda(d-a)}{4\pi^2(x-b)^2} - \frac{\sqrt{\lambda d}}{\pi x}\cos\left(\frac{kx^2}{2d} + \frac{\pi}{4}\right) - \frac{\sqrt{\lambda(d-a)}}{\pi(x-b)}\cos\left(\frac{k(x-b)^2}{2(d-a)} + \frac{\pi}{4}\right) + \frac{\lambda\sqrt{d(d-a)}}{2\pi^2x(x-b)}\cos\frac{k}{2}\left(\frac{x^2}{d} - \frac{\overline{x-b}^2}{d-a}\right).$$

Of these six terms, the second, third and sixth are small compared with the first, third and fourth and may in calculating the position of the maxima and minima, be left out of account.

To test the results obtained from the formula, measurements have been made of the position of the maxima and minima in several cases. The experimental data and the theoretical values are shown for comparison in the table, the latter being taken from the illumination curves plotted from the formula. Fig. 4 shows three illumination curves and illustrates the general effect of the superposition of the fringes due to two edges.

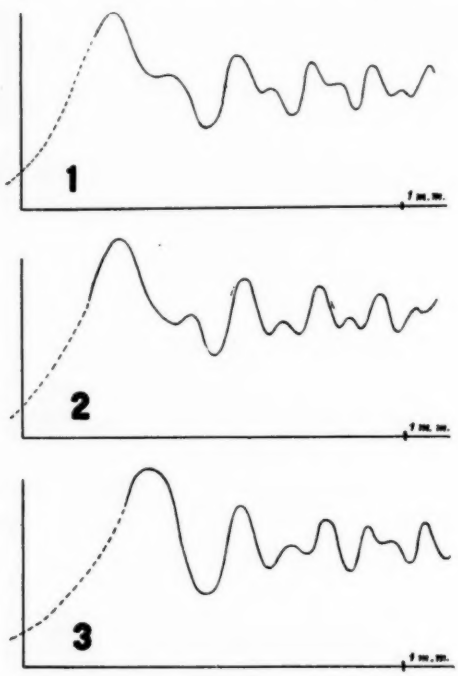


Fig. 4.

Illumination curves for two edges.

1. $b.$ = .000 G.m. 2. $b.$ = .002 G.m. 3. $b.$ = .014 G.m.

I would like to express my indebtedness to Professor C. V. Raman for the help and encouragement he gave me during the course of this investigation, which was carried on last summer in the Laboratory of the Indian Association for the Cultivation of Science, Calcutta.

Positions of the maxima and minima of illumination (measured from the geometrical edge of the shadow of the edge nearest the plane of observation.

1. $d = 20$ cm., $a = 10$ cm. and $b = .002$ cm.

Bands.	Observed.	Calculated.
First maximum	+.024	.025
First minimum	+.040	.040
Second maximum	+.046	.044
Second minimum	+.052	.050
Third maximum	+.058	.058
Third minimum	+.065	.065
Fourth minimum	+.073	.073
Fifth minimum	+.083	.083
Sixth minimum	.....	.089
Seventh minimum	+.096	.098
Eighth minimum	+.108	.105

2. $d = 20$ cm.	$a = 10$ cm.	$b = .014$ cm.
First minimum	+.0378	.034
Second minimum	+.0526	.051
Third minimum	+.0617	.060
Fourth minimum	+.0712	.072
Fifth minimum	.....	.081
Sixth minimum	+.0892	.088
Seventh minimum	+. 102	...

3. $d = 20$ cm.	$a = 10$ cm.	$b = .022$ cm.
First minimum	.021	.019
Second minimum	.032	.032
Third minimum	.042	.042
Fourth minimum	.051	.050
Fifth minimum	.059	.059

The agreement is fairly satisfactory.

Queen's College,
Benares, India,
March 28, 1918.

ON THE MECHANICAL AND ELECTRODYNAMICAL PROPERTIES OF THE ELECTRON.

BY MEGH NAD SAHA.

THE object of the present paper is to extend Minkowski's method[1] of four-dimensional analysis to the investigation of the mechanical and electrodynamical problems connected with the electron. As is well-known, Minkowski's four-dimensional analysis is based on the principle of relativity, and we have thereby to abandon two time-honored concepts of physics, *i. e.*, absolute independence of time and space, and the constancy of mass. The correctness of these two principles is no longer a matter of hypothesis, but is founded on experiments. It is therefore to be hoped that the results of these investigations will be helpful to us for the elucidation of the mechanical and electrical problems connected with the electron, though sometimes difficulty may be encountered in putting proper interpretation on these results.

The notation is the same as that adopted by Minkowski, and for the convenience of the reader, it is explained at the very outset.

I.

$(x, y, z, l \pm ict)$ denotes the space and time coördinates of any point in the four-dimensional world

$$(\omega_1, \omega_2, \omega_3, \omega_4) = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}\left[\frac{u_1}{c}, \frac{u_2}{c}, \frac{u_3}{c}, i\right]$$

denotes the velocity four-vector of the point.

We put

$$ds^2 = -(dx^2 + dy^2 + dz^2 + dl^2)$$

therefore we have

$$(\omega_1, \omega_2, \omega_3, \omega_4) = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}, \frac{dl}{ds}\right), \text{ and } \sqrt{-}\,(\omega_1, \omega_2, \omega_3, \omega_4)$$

[1] Minkowski's method of four-dimensional analysis is expounded in two papers: (1) Raum und Zeit, published in the Phys. Zeits., and (2) Die Grundgleichungen für die Electro-magnetischen Vorgänge in bewegten Körpern-Gött. Nach, 1908. These two papers have been translated by me, and are being published by the Calcutta University.

denote the direction cosines of the four-dimensional tangent to the path of the particle. c = velocity of light in space.

We put

$$(f_{23}, f_{31}, f_{12}) = (Hx, Hy, Hz),$$

the components of the magnetic field, and

$$(f_{14}, f_{24}, f_{34}) = -i[Ex, Ey, Ez],$$

the components of the electric field. Minkowski has shown that f constitutes a six-vector.

$$(a_1, a_2, a_3, a_4) = [F, G, H, l\phi],$$

are the components of the potential four-vector; (F, G, H) are the vector potentials Φ is the scalar potential.

ρ = electrical space-density;

$$\rho\left[\frac{u_1}{c}, \frac{u_2}{c}, \frac{u_3}{c}, i\right] = \rho_0(\omega_1, \omega_2, \omega_3, \omega_4)$$

are the components of the stream four-vector s;

$$\rho_0 = \rho\sqrt{1 - \frac{u^2}{c^2}}$$

is known as the rest-density of electricity.

The vector operator

$$\square = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} + \frac{\partial}{\partial l}\right),$$

is known as the lor and the scalar operator

$$\square^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial l^2}\right)$$

is known as the generalized D'alembertian.

The equations of electrodynamics can be written in the forms

$$\text{lor } f = s, \qquad \text{lor } f^* = 0,$$

$$f = \text{Curl } a; \qquad \square^2 a = -s, \qquad \square a = 0.$$

2. The Scalar and Vector Potentials of a Moving Electron.

Lienard,[1] and almost simultaneously Wiechert[2] showed that the scalar and vector potentials are given by the expressions

$$\phi = -\frac{e}{r\left(1 - \frac{ur}{c}\right)}, \qquad (F, G, H) = -\frac{e(u_1, u_2, u_3)}{cr\left(1 - \frac{ur}{c}\right)}, \qquad (1)$$

[1] Lienard, L'eclairage electrique, 16 (1898), pp. 5, 53 and 106.

[2] Wiechert, Arch. Neérl., (2), 5 (1900).

If P be the point at which the potentials are calculated at the time t and M be the position of the electron at the time t_0, where $MP = c(t-t_0)$, the distance MP is denoted by r and $[u]$ denotes the velocity in the position M, and (ur) its component in the direction of r.

The formulæ are deduced from the theory of retarded potential and do not involve the principle of relativity.

Several investigators[1] have shown that the formulæ can also be deduced from the theory of relativity and can be thrown into the compact form

$$a = \frac{e[\omega]}{[R\omega]},$$

R being the four-vector joining the two points, $[R\cdot\omega]$ denoting the scalar product of R and ω.

It is quite clear that the forms (1) and (2) are quite equivalent.

In a paper published elsewhere, it has been shown that from Mimkowski's four-dimensional analysis we obtain

$$a = \frac{e[\omega]}{P}. \tag{3}$$

In this formula, (x, y, z, l) denote the time-space coördinates of the electron (A), $(\omega_1, \omega_2, \omega_3, \omega_4)$ its velocity-components, $(x'\ y'\ z'\ l')$ denote the space-time coördinates of the point B at which the potentials are estimated,

P denotes the four-dimensional perpendicular distance of B from the axis of motion of (A); since the direction-cosines of this axis are $-i\ (\omega_1, \omega_2, \omega_3, \omega_4)$, we have

$$\begin{aligned} P^2 &= (x-x')^2 + (y-y')^2 + (z-z')^2 + (l-l')^2 \\ &\qquad + [(x-x')\omega_1 + (y-y')\omega_2 + (z-z')\omega_3 + (l-l')\omega_4]^2 \\ &= R^2 + [R\omega]^2. \end{aligned}$$

Now if we make the assumption that the time coördinates are so chosen that

$$R^2 = (x-x')^2 + (y-y')^2 + (z-z')^2 + (l-l')^2 = 0$$

i. e.,

$$c^2(t-t')^2 = r^2, \tag{4}$$

i. e.,

$$c(t-t') = r,$$

the formula (3) becomes the same as (2) and therefore (1). Also the assumption which we make here about the interval between the time-coördinates is identical with the premises of Lienard and Wiechert.

[1] Sommerfeld, Über die Relativitäts-theorie, Ann. d. Physik, Vols. 32 and 33.

I am not quite certain whether this assumption (4) which is made here is at all essential. I am inclined to think that it is not essential, but necessary only for the interpretation of the result to those three-dimensional beings whose senses are not sharpened enough to enable them to grasp a result expressed in four-dimensional figures.

3. The Electric and Magnetic Fields Due to a Moving Electron.

If a denote the potential four-vector, the components of the six-vector f giving the electric and magnetic fields are given by

$$f = \text{Curl } a = \begin{vmatrix} \frac{\partial}{\partial x'} & \frac{\partial}{\partial y'} & \frac{\partial}{\partial z'} & \frac{\partial}{\partial'} \\ a_1 & a_2 & a_3 & a_4 \end{vmatrix}.$$

Thus

$$f_{12} = \frac{\partial a_2}{\partial x'} - \frac{\partial a_1}{\partial y'} = -\frac{e}{P^3}[\omega_3\alpha_2 - \omega_2\alpha_3], \text{ etc.} \tag{5}$$

where

$$\alpha_1 = P\frac{\partial P}{\partial x'}, \quad \alpha_2 = P\frac{\partial P}{\partial y'} \ldots \text{ etc.}$$

we can easily verify that if we put $c(t - t') = r$, we have

$$Hx = \frac{e\beta^2}{r^3\lambda^3}\left[\frac{u_2}{c}(z - z') - \frac{u_3}{c}(y - y')\right],$$

where

$$\lambda = \left(1 - \frac{Ur}{c}\right),$$

$$= \frac{e\beta^2}{r^3\lambda^3}[\mathbf{u} \times \mathbf{r}]. \tag{6}$$

The electric forces are given by

$$f_{14} = -iEx = \frac{\partial a_4}{\partial x'} - \frac{\partial a_1}{\partial l'},$$

$$= -\frac{e}{P^3}[\omega_1(l - l') - \omega_4(x - x')],$$

$$= -i\frac{e\beta^2}{r^3\lambda^3}\left[r\frac{u_1}{c} - (x - x')\right], \tag{7}$$

and generally

$$Ex = \frac{e\beta^2}{r^3\lambda^3}\left[(x - x') - \frac{ru_1}{c}\right], \qquad E = \frac{e\beta^2}{r^3\lambda^3}\left[\mathbf{r} - \frac{r\mathbf{u}}{c}\right].$$

These values are widely different and simpler than the values obtained

from the older theories, for example, compare the values given by Crehore.[1]

The discrepancy is due to the fact that in these older theories, we always assume that the equation

$$(x - x')^2 + (y - y')^2 + (z - z')^2 + (l - l')^2 = 0,$$

is an essential condition. But in performing differentiations with regard to (x', y', z', l') we here assume that they are quite independent of (x, y, z, l). I am not quite definite as to which of these two standpoints is correct but I am inclined to think that my standpoint is more in accordance with Minkowski's ideas of time and space. However it is preferable to keep an open mind on this point.

3. MAXWELL'S STRESSES, POYNTING-VECTOR, ETC.

Minkowski has shown that if we multiply f by its own matrix, we obtain a matrix

$$ff = \begin{vmatrix} S_{11} - L^2 & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} - L^2 & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} - L^2 & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} - L^2 \end{vmatrix}$$

where

$$S_{11} = \tfrac{1}{2}[f_{23}^2 + f_{34}^2 + f_{42}^2 - f_{12}^2 - f_{13}^2 - f_{14}^2],$$

$$S_{12} = [f_{13}f_{32} + f_{14}f_{42}],$$

$$L = \tfrac{1}{2}[f_{23}^2 + f_{32}^2 + f_{12}^2 + f_{14}^2 + f_{24}^2 + f_{34}^2],$$

and the matrix

$$\frac{1}{4\pi}\begin{vmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{vmatrix} = \begin{vmatrix} X_x & X_y & X_z \\ Y_x & Y_y & Y_z \\ Z_x & Z_y & Z_z \end{vmatrix}$$

denote the Maxwellian stresses, $i(S_{14}, S_{24}, S_{34})$ denote the components of the Poynting vector, and S_{44} is the energy function. We have generally

$$X_x = \frac{1}{8\pi}[f_{23}^2 + f_{34}^2 + f_{42}^2 - f_{12}^2 - f_{13}^2 - f_{14}^2], \qquad (8)$$

$$X_y = \frac{1}{4\pi}[f_{13}f_{32} + f_{14}f_{42}],$$

etc.

Now on the standpoint taken up by me, it is quite easy to calculate these qualities. It can be shown that

$$X_x = \frac{e^2}{8\pi P^2}[-P^2(1 + 2\omega_1^2) + \alpha_1^2], \quad X_y = \frac{e^2}{8\pi P^6}[-\omega_1\omega_2 P^2 + \alpha_1\alpha_2], \qquad (9)$$

[1] PHYS. REV., July, 1917, p. 448.

The Poynting-vector

$$(X_1, Y_1, Z_1) = \frac{e^2}{8\pi P^6}[(-\omega_1\omega_4 P^2 + \alpha_1\alpha_4), \quad (-\omega_2\omega_4 P^2 + \alpha_2\alpha_4), \quad (-\omega_3\omega_4 P^2 + \alpha_3\alpha_4)]$$

and the energy function

$$S_{44} = L = \frac{e^2}{8\pi P^6}[-P^2(1 + 2\omega_4^2) + \alpha_4^2]$$

where

$$\alpha_1 = P\frac{\partial P}{\partial x'}, \quad \alpha_2 = P\frac{\partial P}{\partial y'}, \quad \text{etc.}$$

and

$$\alpha^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 = P^2.$$

4. THE LAW OF ATTRACTION BETWEEN TWO MOVING ELECTRONS.

We can now proceed to find out the attraction which one moving electron exerts upon another moving electron.

According to Lorentz's theorem the components of the force acting on an electron (A) moving in any electromagnetic field are

$$\begin{aligned} X &= e[\omega_2 f_{12} + \omega_3 f_{13} + \omega_4 f_{14}], \\ Y &= e[\omega_1 f_{21} + \omega_3 f_{23} + \omega_4 f_{24}], \\ Z &= e[\omega_1 f_{31} + \omega_2 f_{32} + \omega_4 f_{34}], \end{aligned} \tag{10}$$

and we can also add the fourth or the time component

$$L = -\frac{ie}{c}[Xu_1 + Yu_2 + Zu_2], \quad \beta = \sqrt{1 - \frac{v^2}{c^2}}$$

which is proportional to the rate at which work is done by the moving charge,—we have

$$L = e[\omega_1 f_{41} + \omega_2 f_{42} + \omega_3 f_{43}].$$

In this case, the field is due to the second electron (charge e', position $x'\ y'\ z'\ l'$, velocity components $\omega_1'\ \omega_2'\ \omega_3'\ \omega_4'$).

According to the last section, the potential four-vector

$$a = \frac{e'[\omega']}{P'}, \quad \text{where} \quad P'^2 = R^2 + [R\omega']^2.$$

We have now, since $f = \text{Curl } a$,

$$X = ee'\left[\omega_2\left\{\frac{\partial}{\partial x}\left(\frac{\omega_2'}{P'}\right) - \frac{\partial}{\partial y}\left(\frac{\omega_1'}{P}\right)\right\} + \omega_3\left\{\frac{\partial}{\partial x}\left(\frac{\omega_3'}{P'}\right) - \frac{\partial}{\partial z}\left(\frac{\omega_1'}{P'}\right)\right\} + \omega_4\left\{\frac{\partial}{\partial x}\left(\frac{\omega_4'}{P'}\right)\frac{\partial}{\partial l}\left(\frac{\omega_1'}{P'}\right)\right\}\right.$$

$$\left. = ee'\left[\frac{\partial}{\partial x}\left(\frac{\omega_1\omega_1' + \omega_2\omega_2' + \omega_3\omega_3' + \omega_4\omega_4'}{P'}\right) - \left(\omega_1\frac{\partial}{\partial x} + \omega_2\frac{\partial}{\partial y} + \omega_3\frac{\partial}{\partial z} + \omega_4\frac{\partial}{\partial l}\right)\left(\frac{\omega_1'}{P'}\right)\right],\right.$$

Now putting

$$\Phi = ee'(\omega_1\omega_1' + \omega_2\omega_2' + \omega_3\omega_3' + \omega_4\omega_4')/P',$$

we find that

$$X = \frac{\partial\Phi}{\partial x} - \frac{d}{ds}\left(\frac{\partial\Phi}{\partial\frac{dx}{ds}}\right),$$

where $\partial/\partial x$ denotes differentiation in which x is explicitly involved, similarly with

$$\frac{\partial}{\partial\omega_1} = \frac{\partial}{\partial\frac{dx}{ds}}.$$

$$\frac{d}{ds} = \left(\omega_1\frac{\partial}{\partial x} + \omega_2\frac{\partial}{\partial y} + \omega_3\frac{\partial}{\partial z} + \omega_4\frac{\partial}{\partial l}\right),$$

as is easily seen. We have similarly

We have similarly

$$Y = \frac{\partial\Phi}{\partial y} - \frac{d}{ds}\left(\frac{\partial\Phi}{\partial\frac{dy}{ds}}\right), \qquad Z = \frac{\partial\Phi}{\partial z} - \frac{d}{ds}\left(\frac{\partial\Phi}{\partial\frac{dz}{ds}}\right),$$

$$L = \frac{\partial\Phi}{\partial l} - \frac{d}{ds}\left(\frac{\partial\Phi}{\partial\frac{dl}{ds}}\right). \tag{11}$$

We can say that Φ is the kinetic-potential of the electron (A) in the field of the electron (B). Similarly if Φ' denote the Kinetic-Potential of the electron (B) in the field of (A),

$$\Phi' = ee'(\omega_1\omega_1' + \omega_2\omega_2' + \omega_3\omega_3' + \omega_4\omega_4')/P,$$

$$P^2 = R^2 + (R\omega)^2.$$

and we have similarly

$$X' = \frac{\partial\Phi'}{\partial x'} - \frac{d}{ds'}\left(\frac{\partial\Phi'}{\partial\frac{dx'}{ds'}}\right). \tag{12}$$

Let us now interpret the results in three dimensions. We have

$$X = \frac{ee'\beta'^2}{r^3\lambda^3\beta'}(x - x')\left(1 - \frac{uv\cos\theta}{c^2}\right) - \frac{ee'\beta'^2}{r^2\lambda^3\beta c}\left(1 - \frac{Ur}{c}\right)u_1', \tag{13}$$

where

$$\beta = \sqrt{1 - \frac{v^2}{c^2}}, \qquad \beta' = \sqrt{1 - \frac{v'^2}{c^2}}, \qquad \lambda = \left(1 - \frac{Ur'}{c}\right).$$

In three dimensions, the forces are equivalent to a force of repulsion

$$\frac{ee'\beta'^2}{r^3\lambda^3\beta}\left(1-\frac{uv\cos\theta}{c^2}\right)\mathbf{r},$$

in the direction of the line joining the two points, and a force

$$\frac{ee'\beta'^2}{r^2\lambda^3\beta c}\left(1-\frac{Ur}{c}\right)\mathbf{u}', \tag{14}$$

in the direction of the velocity of the second or the attracting point.

We thus perceive that the force which comes out in a very simple form in four dimensions takes a very complicated form in three dimensions.

The kinetic potential

$$\phi=\frac{ee'\left(1-\frac{vv'\cos\theta}{c^2}\right)}{r\left(1-\frac{U'r}{c}\right)\beta}. \tag{15}$$

This kinetic potential is practically coincident with the kinetic potential assumed by Clausius in order to find out the law of attraction between two moving charges of electricity; Clausius has shown that this kinetic potential leads us to the celebrated electrodynamic laws of Ampere. A short résumé of the work done in this connection is given below for the purpose of comparison. The problem was first enunciated by Gauss in the year 1835, and was called by him the fundamental keystone of electrodynamics.[1]

1. Gauss (1835): The forces are the derivatives with regard to (x, y, z) of the potential function,

$$\phi=\frac{ee'}{r}\left(1-\frac{3}{2c}\frac{d^2r}{dt^2}\right).$$

2. Weber (1843) takes the potential function

$$\phi=\frac{ee'}{r}\left(1-\frac{1}{c^2}\frac{dr^2}{dt}\right).$$

Both of these forms have been long discredited. Later writers have pointed out that the force cannot be simply the derivations with regard to (x, y, z) of some potential function, but are the Lagrangian derivatives of a certain kinetic-potential. We give the form of this kinetic potential according to different investigators.

[1] For the literature on the subject, see Maxwell, Electricity and Magnetism, Vol. 2, Chap. XXIII., and J. J. Thomson, Application of Dynamics to Problems of Physics and Chemistry, pp. 35.

1. Clausius (1881);

$$\phi = \frac{ee'}{r}\left(1 - \frac{uu' \cos\theta}{c^2}\right),$$

where v and u' are the velocities of the two electrons, and θ is the angle between their lines of motion. In two papers communicated to the Crelle's journal,[1] Clausius deduces Ampere's laws of electrodynamical action between two currents from this law.

2. J. J. Thomson

$$\phi = \frac{ee'}{r}\left(1 - \frac{\mu}{3c^2} uu' \cos\theta\right) \qquad (\mu = \text{magnetic permeability} = 1).$$

Crehore[2] has calculated the forces components according to J. J. Thomson's theory.[3] He finds that the forces are equivalent to

$F_1 = \frac{ee'}{r^2}$, a repulsion along the line joining the centers.

$F_2 = \frac{ee'}{c^2 r^2} uu' \cos\theta$, an attraction along the same line.

$F_3 = \frac{ee'}{r} u'$, a force in the direction opposite to the acceleration of the second charge.

$F_4 = ee'u' \frac{d}{dt}\left(\frac{1}{r}\right)$, a force in a direction opposite to the velocity of the second charge.

3. Sommerfeld[4] has also calculated the ponderomotive forces, assuming that the value of the potential four-vector

$$a = \frac{e[\omega]}{(R\omega)},$$

and using the condition

$$(x - x')^2 + (y - y')^2 + (z - z')^2 + (l - l')^2 = 0$$

in course of differentiation. Their forms are a bit too complicated.

5. Equations of Motion of the Electron.

Minkowski[5] deduces the equations of motions of a ponderable particle by means of a variational process in which the function

$$\int m_0 c^2 ds, \text{ where } ds^2 = -(dx^2 + dy^2 + dz^2 + dl^2) = c^2 dt^2\left(1 - \frac{u^2}{c^2}\right)$$

is used instead of the three-dimensional $\int T dt$.

[1] Vols. 82 and 83.

[2] It will be seen that forces F_1, F_2, F_4 are, but for some minor details, represented in our formula. Force F_3 does not occur at all.

[3] Phil. Mag., 1913.

[4] Ann. d. Phys., vols. 32 and 33, Über die Relativitäts theories.

[5] Minkowski, *loc. cit.* Anhaup, Mechanics.

He obtains

$$m_0c^2\frac{d^2x}{ds^2}=X,\qquad m_0c^2\frac{d^2y}{ds^2}=Y,\qquad m_0c^2\frac{d^2z}{ds^2}=Z,\qquad m_0c^2\frac{d^2l}{ds^2}=L. \tag{16}$$

Now we have

$$X=e[\omega_2 f_{12}+\omega_3 f_{13}+\omega_4 f_{14}],$$

according to Lorentz's theorem. We have also

$$\begin{aligned}\frac{d^2x}{ds^2}&=\left(\omega_1\frac{\partial}{\partial x}+\omega_2\frac{\partial}{\partial y}+\omega_3\frac{\partial}{\partial z}+\omega_4\frac{\partial}{\partial l}\right)\omega_1\\&=\omega_1\frac{\partial}{\partial x}[\tfrac{1}{2}(\omega_1^2+\omega_2^2+\omega_3^2+\omega_4^2)]+\omega_2\left[\left(\frac{\partial\omega_1}{\partial y}-\frac{\partial\omega_2}{\partial x}\right)\right.\\&\qquad\left.+\omega_3\left(\frac{\partial\omega_1}{\partial z}-\frac{\partial\omega_3}{\partial x}\right)+\omega_4\left(\frac{\partial\omega_1}{\partial l}-\frac{\partial\omega_4}{\partial x}\right)\right]\\&=-(\omega_2\Omega_{12}+\omega_3\Omega_{13}+\omega_4\Omega_{14}),\text{ putting }\Omega_{hk}=\frac{\partial\omega k}{\partial xh}-\frac{\partial\omega h}{\partial xk}.\end{aligned}$$

Hence we have the four equations, putting $\mu=c^2m_0/e$,

$$\left.\begin{aligned}&\omega_2(f_{12}+\mu\Omega_{12})+\omega_3(f_{13}+\mu\Omega_{13})+\omega_4(f_{14}+\mu\Omega_{14})=0\\&\omega_1(f_{21}+\mu\Omega_{21})+\omega_3(f_{23}+\mu\Omega_{23})+\omega_4(f_{24}+\mu\Omega_{24})=0\\&\omega_1(f_{31}+\mu\Omega_{31})+\omega_2(f_{32}+\mu\Omega_{32})+\omega_4(f_{34}+\mu\Omega_{34})=0\\&\omega_1(f_{41}+\mu\Omega_{41})+\omega_2(f_{42}+\mu\Omega_{42})+\omega_3(f_{43}+\mu\Omega_{43})=0\end{aligned}\right\}.$$

Of these, only three are independent; the fourth can be deduced from the first three.

We have now identically

$$\begin{vmatrix}&f_{12}+\mu\Omega_{12}&f_{13}+\mu\Omega_{13}&f_{14}+\mu\Omega_{14}\\f_{21}+\mu\Omega_{21}&&f_{23}+\mu\Omega_{23}&f_{24}+\mu\Omega_{24}\\f_{31}+\mu\Omega_{31}&f_{32}+\mu\Omega_{32}&&f_{34}+\mu\Omega_{34}\\f_{41}+\mu\Omega_{41}&f_{42}+\mu\Omega_{42}&f_{43}+\mu\Omega_{43}&\end{vmatrix}=0,$$

i. e.,

$$\begin{aligned}&(f_{12}+\mu\Omega_{12})(f_{34}+\mu\Omega_{34})+(f_{23}+\mu\Omega_{23})(f_{14}+\mu\Omega_{14})\\&\qquad+(f_{31}+\mu\Omega_{31})(f_{24}+\mu\Omega_{24})=0.\end{aligned} \tag{17}$$

The condition is evidently satisfied if we put

$$-\mu=\frac{f_{12}}{\Omega_{12}}=\frac{f_{23}}{\Omega_{23}}=\frac{f_{31}}{\Omega_{31}}=\frac{f_{14}}{\Omega_{14}}=\frac{f_{24}}{\Omega_{24}}=\frac{f_{34}}{\Omega_{34}}. \tag{18}$$

If of these equations, any three are satisfied the remaining three come out automatically from the equations of motion. But we cannot possibly be sure of the authenticity of these relations unless it can be deduced from an independent source. For this purpose let us take the original variational equations.

Let (X, Y, Z, L) represent the components of the force four-vector at any point, which is subjected to a virtual displacement $\delta x, \delta y, \delta z, \delta l$.

Then

$$\delta W = X\delta x + Y\delta y + Z\delta z + L\delta l,$$

i. e., if we call

$$W = \frac{\partial A}{\partial s}, \qquad A = \int W ds,$$

$$\begin{aligned}\delta A = \int \delta W ds &= \int (X\delta x + Y\delta y + Z\delta z + L\delta l) ds \\ &= e\int [f_{12}(dy\delta x - \delta y dz) + f_{23}(dz\delta y - \delta z dy) + f_{31}(dx\delta z - \delta x dz) \\ &\qquad + f_{14}(dl\delta x - \delta l dx) + f_{24}(dl\delta y - \delta l dy) + f_{34}(dl\delta z - \delta z dl)].\end{aligned}$$

Now the function $\int m_0 c^2 ds$ can also be subjected to a variational process. Since

$$ds = \omega_1 dx + \omega_2 dy + \omega_3 dz + \omega_4 dl,$$

we find

$$\begin{aligned}\delta \int m_0 c^2 ds = - m_0 c^2 \int [&\Omega_{12}\delta S_{12} + \Omega_{23}\delta S_{23} + \Omega_{31}\delta S_{31} \\ &+ \Omega_{14}\delta S_{14} + \Omega_{24}\delta S_{24} + \Omega_{34}\delta S_{34}],\end{aligned}$$

where

$$\delta S_{kk} = dx_k \delta x_k - \delta x_k dx_k.$$

Thus from the variational equation

$$\delta \int m_0 c^2 ds + \int \delta W . ds = 0;$$

i. e., from the principle of least action, keeping the initial and final points fixed, we obtain the original equation

$$\int [(f_{12} + \mu\Omega_{12})\delta S_{12} + \ldots] = 0.$$

The relations (18) thus seem to be borne out by independent evidence.

Difficulty is encountered here about the interpretation of the terms $(\Omega_{12}, \Omega_{23}, \ldots)$ in three dimensions Ω is evidently a six-vector being the four-dimensional curl of the velocity four-vector. The components $[\Omega_{23}, \Omega_{31}, \Omega_{12}]$ are evidently connected with rotations

$$\left[\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}, \quad \frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}, \quad \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right]$$

and $[\Omega_{14}, \Omega_{24}, \Omega_{34}]$ are connected with the accelerations

$$\left[\frac{d^2x}{dt^2}, \quad \frac{d^2y}{dt^2}, \quad \frac{d^2z}{dt^2}\right]$$

but the exact interpretation in three dimensions has not yet been obtained. We can style Ω as the acceleration six-vector.

On a future occasion, I hope to communicate the result of my investigations on the orbits of the electron under different conditions.

In conclusion, I wish to express my best thanks to my friend, Mr. S. N. Basu, for much help, and useful criticism.

UNIVERSITY COLLEGE OF SCIENCE,
CALCUTTA, INDIA,
May 11, 1918.

A NEW EXPERIMENTAL DETERMINATION OF THE BRIGHTNESS OF A BLACK BODY, AND OF THE MECHANICAL EQUIVALENT OF LIGHT.

By Edward P. Hyde, W. E. Forsythe and F. E. Cady.

Part I. Brightness of a Black Body as a Function of its Temperature.

THE brightness or candlepower per cm² of a black body at various temperatures has long been a matter of interest. Lummer and Kurlbaum,[1] in connection with their work on radiation, determined the relation between brightness and temperature for radiating platinum and expressed the opinion that the exponential relation found would hold at least approximately for a black body. Subsequently[2] Lummer and Pringsheim determined the brightness of a black body at three temperatures (1448° K., 1598° K. and 1708° K.) and extrapolated their observed curve to give the brightness up to 2073° K., at which temperature they ascribed the brightness of 100 Hefner candles per cm².

A couple of years later Nernst,[3] starting from one of the extrapolated points given by Lummer and Pringsheim, found a brightness of 91 Hefner candles per cm² at the temperature of melting platinum (taken by him as 2018° K.), and a brightness of 1210 Hefner candles per cm² at the temperature of melting iridium, which latter he calculated to be 2621° K. Subsequently[4] Nernst undertook a new set of measurements on the brightness of a black body, carrying his determinations over a range of temperature extending from about 1460° K. to about 2280° K. His temperature measurements of the black body were made with a Wanner pyrometer, and were based on a temperature of 1337° K. as the melting point of gold, and a value of 14600 μ deg. for the constant C_2 in the Planck equation.

From the observed data Nernst computed the values of the constants A and B in the equation

$$\log K = -\frac{A}{T} + B,$$

[1] Verh. d. Deut. Phys. Gesel., 2, p. 90, 1900.
[2] Phys. Zeit., 3, p. 97, 1901.
[3] Ibid., 4, p. 733, 1903.
[4] Ibid., 7, p. 380, 1906.

which had first been given by Rasch[1] and subsequently deduced on what Nernst considers more justifiable theoretical grounds by Haber[2] and Lucas[3] as expressing the relation between brightness, K, and temperature, T. Nernst apparently relies on the values deduced from this equation as more accurate than the individual direct determinations. With the use of this equation he computes the brightness of the black body at the temperature of melting platinum, taken as 2018° K., to be 63.4 Hefner candles per cm², rather than 91 as given in his original paper. He offers an explanation of the cause of error in the earlier work.

It should be noted in passing that Rasch's equation is deducible from the Planck equation as the variation in brightness in some one wave-length, and so is the theoretical foundation for the method of heterochromatic photometry proposed some time earlier by Crova.[4] But it has been recognized for a long time that Crova's method is only approximate over more than a small temperature interval, since the effective wave-length varies with the temperature.[5]

Eisler[6] in 1904, assuming Planck's equation for the distribution of energy in the spectrum of a black body and using Langley's original data on visibility computed the relative brightnesses of a black body at different temperatures, obtaining a relation of brightness with temperature from which he computed the exponent x in the equation

$$\frac{H_1}{H_2} = \left(\frac{T_1}{T_2}\right)^x,$$

where H_1 and H_2 are the relative brightnesses at the two respective temperatures T_1 and T_2 taken close together. The curve of x as a function of temperature passes fairly closely through the points plotted from the observations of Lummer and Pringsheim.

Various computations[7] after the method of Eisler have been made in recent years, using visibility data obtained in more accurate ways and unquestionably much nearer the truth than the original data of Langley, but these computations are founded on the original experimental brightness values of Lummer and Pringsheim, and the subsequent values of Nernst, and there is reason to believe that these experimental values are subject to some error, and that, moreover, in each case there is some question about the relative computed brightnesses on account either of

[1] Ann. d. Phys., 14, p. 193, 1904.
[2] Thermodynamik technisches Gasreactionen (Munich), 1905.
[3] Phys. Zeit., 6, p. 19, 1905.
[4] Comptes Rendus, 93, p. 512, 1881.
[5] PHYS. REV., 32, p. 320, 1911.
[6] Elek. Zeit., 25, p. 188, 1904.
[7] Elec. World, 57, p. 1565, 1911. Verh. d. Deut. Phys. Gessell., 17, p. 219, 1915.

the temperature scale or of the visibility data assumed. The only recent experimental data available are those of Langmuir,[1] and those of Ives and Kingsbury.[2] The former conducted his measurements on a tungsten filament, and his results are subject to some modification largely on account of the values assumed for the emissive power of tungsten which more precise determinations have shown to be in error. The experimental results of Ives and Kingsbury[3] determined both by the use of a black-body furnace and by the application of the wedge method in measuring black-body brightness at the melting points of platinum and gold respectively, are not very concordant, and are given but little weight by the authors themselves.

It therefore seemed opportune to the authors to undertake a new series of experimental determinations of black-body brightness as a function of temperature, and to compare these experimental results with the relative values computed on the assumption of the most probable visibility curve and of the Planck equation for spectral energy distribution after the manner of Eisler. Within the past few years much progress has been made in the establishment of the high temperature scale, and black-body furnaces may now be operated up to temperatures of at least 2600° K. under reasonably favorable conditions for brightness determinations, and with a considerable degree of confidence in the precision of the temperature measurements. Of course, the temperature scale assumed plays an important rôle. This question will be dealt with in some detail in a subsequent section.

In like manner much progress has recently been made in determining the visibility curve of the average eye, but, in the opinion of the authors, since all the recent determinations had been made by the flicker method, there seemed a necessity for a determination of visibility by the direct comparison method. The findings of the flicker method in the investigations of several observers were not very concordant, and the method itself was open to question. The present investigation was therefore interrupted to make a new determination of the visibility curve by the direct comparison method, as will be discussed in a separate section. Fortunately uncertainties in this function produce only second order errors in the relative brightness curve of a black body, and since the range of temperature that can be employed is not extremely large the errors in the relative computed brightnesses are quite small if the chosen visibility curve even roughly expresses the facts. To what extent these

[1] Phys. Rev., (2), 7, p. 322, 1916.

[2] Phys. Rev., (2), 8, p. 177, 1916.

[3] Since this paper was written, Dr. Ives has published a note in the Journal of the Franklin Institute (186, p. 122, 1918), giving data which seem to verify this original value.

considerations are justified will become apparent in the presentation and discussion of the experimental data.

The second part of the paper deals with the evaluation of the mechanical equivalent of light based upon the observed values of black-body brightness and upon the assumption of the constant σ ($E = \sigma T^4$) in the Stefan-Boltzmann law, and of the visibility curve. In this case errors in the visibility curve enter as first order effects and so the precise knowledge of this function becomes of much greater importance.

Temperature Scale.—The relation between observed brightness and temperature of a black body obviously depends upon the chosen scale of temperature. It is customary, in establishing a temperature scale, to assume certain fixed points, such as the melting points of gold and palladium, and to determine other points of the scale on the assumption of some radiation law such as the Stefan-Boltzmann law or the Planck equation. The former of these is accepted without reservation, but there is some slight question regarding the latter, though it is generally believed to represent the facts and is assumed as the basis of optical pyrometry. The values of certain of the constants entering into these equations are still, however, the subject of much discussion and investigation. This is particularly true of the constant C_2 in the Planck equation

$$E_\lambda = C_1 \lambda^{-5} \frac{1}{\epsilon^{\frac{C_2}{\lambda T}} - 1}.$$

As already stated it is customary to assume certain fixed points, such as the melting-points of gold and palladium as measured with the gas thermometer. The best determinations of these values are those of Day and Sossman,[1] but recent investigations would seem to show that the two melting-point temperatures are not consistent with the most probable value of C_2 in the Planck equation.[2] In order to have a definite temperature scale on which to express measurements of temperature the laboratories of the General Electric Company agreed upon the adoption of the gold point as given by Day and Sossman (1336° K.), and of the value 14350 μ deg. for the constant C_2. On this scale the temperature of melting palladium must be changed from 1823° K., the value found by Day and Sossman, to 1828° K. if the earlier measurements of the present authors[3] are correct, since it was found that the value of Day and Sossman was confirmed only upon the assumption of the value of 14460 μ deg. for C_2, if the gold point was accepted as 1336° K.

[1] Am. Jour. of Sci., (4), 29, p. 93, 1910.
[2] Gen. Elec. Rev., 20, p. 819, 1917.
[3] Astrophys. Jour., 42, p. 300, 1915.

Visibility Data.—As stated in an earlier paragraph, there was much doubt in the minds of the authors regarding the acceptability of the various visibility data available when the present investigation was begun. This doubt arose partly from the inconsistency of the various published results but more largely from an inherent skepticism regarding the applicability of visibility data obtained with the flicker photometer to practical problems in heterochromatic photometry in which the method of direct comparison was employed, as in the present investigation.

It is true that only gross differences in the visibility curve can produce appreciable errors in the computed relative brightnesses of a black body at two temperatures differing by only a few hundred degrees, but uncertainties in the visibility curve produce first-order effects in the evaluation of the mechanical equivalent of light. And when the authors found that even the second-order errors in the computed relative brightnesses of a black body on the assumption of the most recently published visibility data[1] with the flicker photometer, as compared with the experimental values found using the direct-comparison method of photometry, were larger than might readily be justified on the ground of experimental error, it was decided to delay the publication of the present investigation until a new determination of the visibility of radiation, using the direct-comparison method, might be made. The results of this investigation, together with a more complete discussion of the question, have already been published.[2]

The visibility data obtained in that investigation will be used in computing the relative brightnesses of a black body at different temperatures, for comparison with the values experimentally observed, and in evaluating the mechanical equivalent of light. For the extreme red end of the spectrum not included in the investigation referred to, use will be made of the visibility data obtained by two of the present authors in an earlier study[3] of visibility in this region; and for the extreme blue end the data of Hartman[4] obtained also in this laboratory will be employed. The complete visibility curve used is that recommended in appendix II. of the recent paper to which reference has just been made.

Apparatus and Method.—The measurements of the brightness of a black body at various temperatures were carried out with two electrically heated black-body furnaces. Most of the measurements were made with a specially designed water-cooled carbon-tube furnace, the remaining ones being made with a platinum-wound porcelain furnace provided with

[1] Bul. Bur. of Stds., 14, p. 167, 1917.
[2] Astrophys. Jour., 48, Sept., 1918.
[3] Ibid., 42, p. 285, 1915.
[4] Ibid., 47, p. 83, 1918.

a Lummer-Kurlbaum black-body tube. The set-up, with the carbon furnace,[1] is shown in Fig. 1, in which *F* is the carbon furnace supplied with a limiting diaphragm. Three different sizes were used with apertures varying from 4 to 5 mm. in diameter. For details of construction and operation of this furnace the reader is referred to a previous paper by the authors on "Color Temperature Scales for Tungsten and Carbon."[2] *P* is a Lummer-Brodhun contrast photometer of special design, in which the contrast is only about 3.5 per cent. on the average and is graduated in either direction from the center—the two features conducing to higher

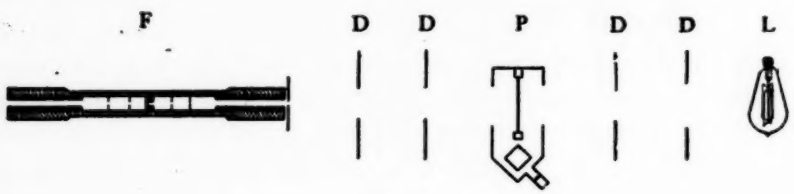


Fig. 1.
Diagrammatic Sketch of Apparatus.

sensibility. *L* is the comparison lamp,—at low temperatures a carbon lamp and at the higher temperatures a vacuum tungsten lamp. A series of black velvet screens *D* with suitable openings shut off stray light from the photometer.

The brightness measurements of the black body were always carried out with the comparison lamp at an approximate color match, except beyond 2400° above which temperature it was considered unsafe to operate the comparison lamp. The calibration of the comparison lamp at the various voltages, involving the difficulties of heterochromatic photometry, was made subsequently. In order to avoid the possible errors arising from individual idiosyncrasies of vision, the candlepower scale chosen was that given for a tungsten lamp by Middlekauff and Skogland.[3] The procedure of calibration consisted in transferring the photometer and comparison lamp to a standard photometer bench, and substituting a tungsten vacuum lamp for the black body. The comparison lamp was then brought successively to the various voltages used against the black body and readings were made against the tungsten lamp. In cases of large color difference blue screens calibrated at the Bureau of Standards were used. The relative candlepowers of the

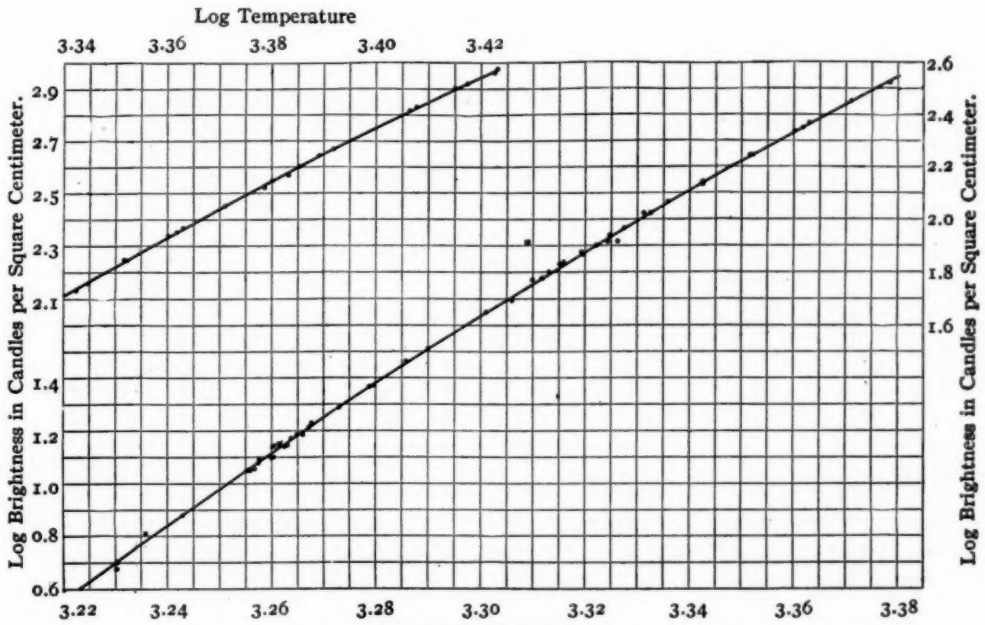
[1] Not as much weight is attached to the results obtained with the platinum-wound furnace because of the photometric difficulties at the low temperatures and with the necessarily small limiting diaphragm which the dimensions of the furnace demanded.

[2] PHYS. REV., (2), 10, p. 395, 1917.

[3] Bul. Bur. of Stds., 11, p. 483, 1915.

standard tungsten lamp were computed from the data of Middlekauff and Skogland, and hence the candlepowers of the comparison lamp, and so the brightnesses of the black body were obtained on this same scale.

The justification of the Middlekauff and Skogland candlepower scale as representative of the average eye may be arrived at through the intercomparison of the visibility curves of several members of this group with the average of the 125 observers employed in the investigation of Coblentz on the visibility of radiation. Such a comparison indicates that the mean of the eight observers employed by Middlekauff and Skogland is not markedly different from the mean of the 125 observers used by Coblentz. Other lines of reasoning would indicate that the scale of Middlekauff and Skogland is not far from correct, as representing an average eye, but that if any change were made it would be in the direction of raising the upper region of the candlepower curve slightly with respect to the curve for the lower temperatures. This change would be less than one per cent. and so, in the absence of more definite information, the Middlekauff-Skogland scale has been adopted in the present investigation.



● Values obtained by computation.
○ Observed values using carbon-tube furnace.
× Observed values using platinum-wound furnace.
□ Values given by Nernst.

Fig. 2.

Brightness of a Black Body at Various Temperatures.

Experimental Results.—The results on black-body brightness expressed in candles per cm² as a function of the temperature in degrees K. are given in a logarithmic form in Fig. 2, and the corresponding numerical results, taken from the curve are given for every 50° in Table I. The circles on the plat are the observed points obtained with the carbon-tube furnace, and the x's are the observed points using the platinum-wound

TABLE I.

Brightness of a Black Body at Various Temperatures.

Temperature.	Brightness Candles per Square Centimeter.	Temperature.	Brightness Candles per Square Centimeter.
1700° K.	5.0_8	2200° K.	$137._6$
1750	7.6_4	2250	177.
1800	11.3	2300	226.
1850	16.3	2350	284.
1900	23.1	2400	354.
1950	32.2	2450	438.
2000	$44._3$	2500	537.
2050	$60._0$	2550	651.
2100	$80._1$	2600	785.
2150	$105._7$	2650	939.

furnace. The curve is drawn according to a least-square solution applied to the 46 observed points and assuming the cubic equation,

$$u = a + bt + ct^2 + dt^3,$$

where u = log candles per cm²,

t = log T − 3.2,

T = temperature in degrees K.

The values of the constants were found to be as follows:

$$a = 0.2626, \qquad c = -12.38,$$

$$b = 14.959, \qquad d = -1.13.$$

The 46 points are so distributed with respect to the least square curve that 23 points lie on one side and 23 on the other, the maximum deviation of any point, as seen from the curve, being about 8 per cent. The two points furthest from the curve by Chauvenet's criterion could be discarded. Thirty points lie within 2.5 per cent. of the curve, the average deviation of all points being about 2.4 per cent. The upper one of the three points obtained by Lummer and Pringsheim is within about 3.5 per cent of the curve (the other two points are at lower temperatures than any included in our observations) and the later observations of Nernst, reduced to the present temperature scale, are indicated by □'s.

The dots are the points computed on the basis of the authors' recently

published data on the visibility of radiation, as discussed above. These computed brightnesses are, of course, only relative, but the value at 2077° K., which is in the neighborhood of the color temperature of the 4-wpc carbon standard lamp, was taken from the least square solution, and the other points were plotted in terms of that. It is seen that the computed points lie on the observed curve within an error of less than 1 per cent. except at the extreme ends of the temperature range. This is better than might be expected owing to the inherent difficulties in the present experiment. Points computed from Coblentz's visibility data show a consistent difference from the observed curve, though the difference is nowhere large.

The accuracy indicated in the brightness measurements suggests the advisability of adopting tentatively the value $70._2$ candles per cm^2 as the brightness of a black body at the temperature 2077° K. and as a value for a primary standard of light. The temperature chosen is approximately the color temperature of the standard 4-wpc carbon lamps.[1]

Part II. The Mechanical Equivalent of Light.

This term has come to be used to mean the watts per lumen of the monochromatic light of greatest visibility, corresponding, in the recent experiments of the authors, to a wave-length $\lambda = 0.556\ \mu$. It is a physical constant of theoretical interest, and also serves as a means of expressing the absolute efficiency of illuminants. It depends on the brightness of a black body, or other source with known emissive power, on the temperature scale chosen, also on the constant σ in the Stefan-Boltzmann law ($E = \sigma T^4$), and finally on the curve of relative visibility of radiation,—*i. e.*, provided it is evaluated in the way to be followed in this paper. It may be determined directly by measuring the radiant flux and the luminous flux from some monochromatic source in a wave-length at or near that of maximum visibility. Determinations have been carried out by this method,[2] but in the opinion of the authors the experimental difficulties are too great to justify much confidence in the results.

In the present investigation the mechanical equivalent M, following the usual procedure, has been evaluated in the following way. Let

$B_0 \equiv$ normal brightness of the black body at any temperature T expressed in candles per cm^2.

[1] The data given above are somewhat different from those published in the advance note in the Journal of the Franklin Institute (Jour. of Franklin Inst., 181, p. 420, 1916.) These differences are due to the assumption of a different temperature scale and to the addition of several new observed points which produced a small change in the curve, particularly at the higher temperatures.

[2] Phys. Rev., (2), 5, p. 269, 1915.

$E_\lambda \equiv$ the radiant flux per cm² per unit wave-length at the wave-length λ.
$V_\lambda \equiv$ the relative visibility of radiation in any wave-length λ in terms of the maximum visibility taken equal to unity.

Then

$$B_0 = \frac{1}{\pi M}\int_0^\infty E_\lambda V_\lambda d\lambda \tag{1}$$

or

$$M = \frac{1}{\pi B_0}\int_0^\infty E_\lambda V_\lambda d\lambda. \tag{2}$$

In the present investigation the visibility function is assumed as extending from 0.40 μ in the blue to 0.76 μ in the red, the values assigned in the two extreme ends of the spectrum being taken from the previously published data of two of the present authors[1] in the red, and of Hartman[2] in the blue.[3] Beyond the chosen limits the added luminosity area would be entirely negligible.

The method employed in evaluating the integral in equation (1) was that of summating the product $E_\lambda V_\lambda \Delta\lambda$ for steps $\Delta\lambda = 0.01\ \mu$, and correcting for the errors of summation by the one third rule. The value of E_λ and therefore that of M depends upon the constant C_1 in Planck's equation, or its equivalent σ in the Stefan-Boltzmann equation, since these two constants are definitely related if C_2 is known. The latter is taken as 14350 μ deg., as discussed in the first part of the paper, and for σ the value $5.7 \times 10^{-12}\ \frac{\text{watts}}{\text{cm}^2\ \text{deg}^4}$, as found by Coblentz, has been assumed as the most probable value. On the basis of these values, C_1 comes out as 3.72×10^{-12} watts cm².

TABLE II.

Mechanical Equivalent M for Various Temperatures.

Temperature.	*M* in Watts per Lumen.	Temperature.	*M* in Watts per Lumen.
1700° K.	0.00147_8	2200° K.	$.00149_6$
1800	$.00149_1$	2300	$.00149_7$
1900	$.00149_8$	2400	$.00149_7$
2000	$.00149_8$	2500	$.00150_2$
2100	$.00149_7$	2600	$.00151_1$
Average.....	.....	.....	0.001496

[1] *Loc. cit.*

[2] *Loc. cit.*

[3] The complete visibility curve used is given in Appendix II. of the paper by the present authors on "The Visibility of Radiation," Astrophys. Jour., Sept., 1918.

TABLE III.

Investigators.	Recommended Values of the Mechanical Equivalent of Light in Watts per Lumen.	Visibility Data.	Constants σ in $\frac{\text{Watts}}{\text{Cm.}^2 \text{ Deg.}^4}$ C_2 in μ Deg.	Method.
P. G. Nutting	0.00120 ± 0.00005	Nutting	$\sigma = 5.7 \times 10^{-12}$ (Presumably)	Measurements of total radiant flux and total luminous flux from acetylene. Coblentz' energy curve for acetylene used.
H. E. Ives	0.00160 ± 0.00003	Ives & Kingsbury Curve from equation	$\sigma = 5.7 \times 10^{-12}$	(1) Measurement of total radiant flux and total luminous flux from a monochromatic source. (2) Measurement of radiant flux through luminosity filter and integral luminous flux direct from the source.
	Secondary value 0.00154	Ives & Kingsbury	$\sigma = 5.7 \times 10^{-12}$ $C_2 = 14350$	(3) Measurement of brightness of a platinum-wedge black body at the melting point of platinum taken as 2037° K.
W. W. Coblentz	0.00162 ± 0.00005 Note: If in method (1) the final data of Hyde, Forsythe and Cady were used, the value derived by this method would be increased by approximately 1½ per cent.	Coblentz and Emerson	$C_2 = 14350$ Melting points as given by Day and Sossman	Several methods using observations by himself and others. (1) Computation using preliminary data on brightness of a black body by Hyde, Forsythe and Cady. (2) Direct measurement on monochromatic radiation, by Coblentz, Ives and Kingsbury. (3) Measurements on incandescent lamps by Coblentz and Emerson.
Hyde, Forsythe and Cady	0.00150 ± 0.00005	Hyde, Forsythe and Cady	$\sigma = 5.7 \times 10^{-12}$ $C_2 = 14350$ Melting point of gold 1336° K.	Direct measurement of brightness of carbon and platinum tube black bodies.

Substituting in equation (5) the brightness of the black body at any temperature, as given in the first part of this paper, a value of the mechanical equivalent M for any temperature is found. If the observed and computed curves of black-body brightness are the same throughout,

then the value of M would be found to be the same for all temperatures, as might be expected. The differences between the two curves as presented in the first part of this paper are so small that the values for M computed for every 100°, as given in Table II., are sensibly constant, except at the extreme ends of the curve, the total range being only 2.2 per cent., and the range from 1800° to 2400° K. being only 0.5 per cent.

The average value from the present experiments is given in Table III., in comparison with the published values of other observers. In each case so far as is possible from the published reports, the method used, the visibility curve employed (if any) and the temperature scale involved are given.

SUMMARY.

1. A new set of experimentally determined values of the brightness of a black body from 1700° to 2600° K. are given on the assumption of a temperature scale based upon Planck's equation taking the gold point as 1336° K. and the constant C_2 as 14350 μ deg.

2. These values, as plotted in a curve obtained by a least-square solution, are compared with the relative computed values on the assumption of the visibility curve recently published by the authors.

3. A value of 70.2 candles per cm² as the brightness of a black body at a temperature of 2077° K. (color match with 4-wpc standard carbon lamps) is proposed tentatively as an absolute standard of light.

4. The mechanical equivalent of light for the wave-length of maximum visibility ($\lambda = 0.566\ \mu$) is computed to be 0.00150 ± 0.00005 watts per lumen.

APPENDIX I.

It is frequently desired to know the rate of change of candlepower or of brightness of a black body at any temperature with a change in temperature or a change in watts. Since the data presented in the body of this paper are somewhat different from other published data it seems advisable to compute these coefficients and at the same time to evaluate the Crova wave-length[1] at different points on the brightness-temperature scale.

Differentiating equation (1) for the brightness of a black body with reference to temperature, the following final equation is obtained:

$$\frac{dB_0}{dT}\frac{T}{B_0} = \frac{C_2}{T}\frac{\int_0^\infty \frac{1}{\lambda} E_\lambda V_\lambda d\lambda}{\int_0^\infty E_\lambda V_\lambda d\lambda}. \tag{3}$$

[1] *Loc. cit.*

If now the summation method employed in the body of the paper in computing the mechanical equivalent of light be used the integral in the denominator is at once obtained; and if the respective terms at any temperature are multiplied by the reciprocals of the wave-length and the summation made, the numerator is also obtained. The ratio of these two integrals at any temperature, multiplied by C_2/T gives the desired percentage change in brightness corresponding to a change of 1 per cent. in temperature. These results are shown in Fig. 3. Since

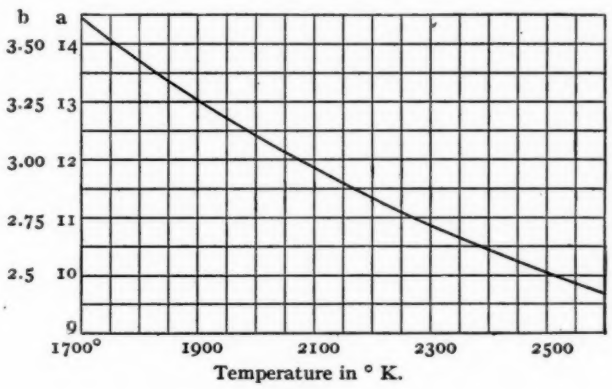


Fig. 3.

Percentage Changes in Brightness of a Black Body at Various Temperatures for (*a*) one per cent. change in temperature and for (*b*) one per cent. change in watts.

$E = \sigma T^4$ expresses the relation between temperature and total watts radiated it follows that if the ordinates of the curve are divided by 4 the resultant curve will show the percentage change of brightness corresponding to a change of 1 per cent. in total watts radiated.

Moreover it follows from equation (3) that if λ_0 is the wave-length such that at some temperature the ratio of the two integrals in equation (3) is $1/\lambda_0$, then at that temperature

$$\frac{dB_0}{dT}\frac{T}{B_0} = \frac{C_2}{\lambda_0 T}, \tag{4}$$

or, the percentage change in brightness corresponding to a change of 1 per cent. in temperature is equal to $C_2/\lambda_0 T$.

But it follows immediately from Wien's equation that

$$\frac{dE_{\lambda T}}{dT} \cdot \frac{T}{E_{\lambda T}} = \frac{C_2}{\lambda T}, \tag{5}$$

or, the percentage change in watts radiated by a black body at any wave-length λ and at any temperature T, corresponding to a change of 1 per

cent. in temperature is $C_2/\lambda T$. Hence if the particular value λ_0 be taken for λ, then from equations (4) and (5) it follows that the percentage change in brightness at any temperature, and the percentage change in watts emitted at that temperature in the wave-length λ_0, corresponding to a change of 1 per cent. in temperature, will be the same. By definition, therefore, λ_0 is the Crova wave-length for that temperature, and hence is readily evaluated since the two integrals in equation (3) are now known. The values of the Crova wave-length thus obtained are given in Fig. 4.

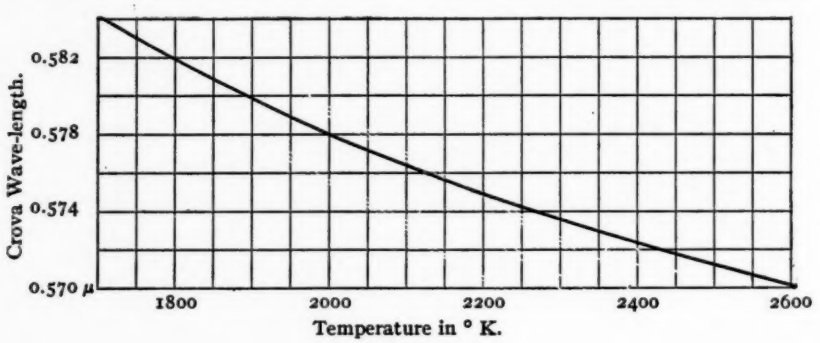


Fig. 4.

Crova Wave-length at Various Temperatures for Black Body Radiation.

It will be seen by comparison with the data published by Kingsbury[1] and by Coblentz[2] that the authors' values for the Crova wave-length are somewhat smaller. This difference is for the most part due to the difference in visibility curves since those obtained by the flicker method show larger values in the longer wave-lengths of the visible spectrum.

NELA RESEARCH LABORATORY,
NATIONAL LAMP WORKS OF GENERAL ELECTRIC CO.,
NELA PARK, CLEVELAND, OHIO,
September, 1918.

[1] PHYS. REV., (2), 7, p. 167, 1916.
[2] Bul. Bur. of Stds., 14, p. 255, 1918.

IONIZATION AND RESONANCE POTENTIALS FOR ELECTRONS IN VAPORS OF ARSENIC, RUBIDIUM, AND CÆSIUM.

BY PAUL D. FOOTE, O. ROGNLEY AND F. L. MOHLER.

WHEN electrons are accelerated in a metallic vapor, at least two types of inelastic impact between an electron and atom occur. The first of these results in an orbital shift of the electrons bound in the atom and the second in the complete removal of an electron or ionization of the atom. The potential differences through which an electron must fall to obtain sufficient velocity or kinetic energy for these two types of collision are known as the resonance and the ionization potentials for the particular metal in question. The resonance and ionization potentials of a series of metals have been determined in this laboratory.[1] The present paper is a report upon the work with arsenic, rubidium and cæsium.

The arrangement of apparatus for arsenic is shown in Fig. 1. Arsenic was vaporized at the bottom of the pyrex glass tube and the vapor, after passing through the superheated ionization chamber, condensed in the upper half of the tube. The ionization chamber was supported entirely by steel rods running from the top of the tube. The source of electrons was a hot tungsten wire cathode, *A*, of low resistance, coated with lime. Surrounding the cathode was a cylinder of iron net, *B*, and outside and coaxial with this a cylinder of sheet iron, *C*. The apparatus was evacuated by means of a Langmuir and Stimson condensation pump to a gas pressure of about 5.10^{-5} cm. Hg as registered by a McLeod gage. Sufficient vapor was obtained by heating the arsenic to about 400 to 500° C.

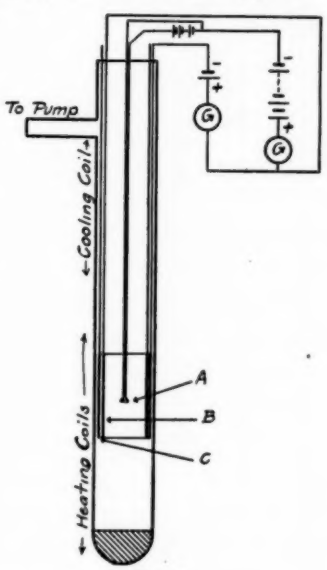


Fig. 1.
Diagram of apparatus used for arsenic.

[1] Na, Tate and Foote, J. Wash. Acad. Sci., 7, p. 517, 1917. Cd, Tate and Foote, Bur. Standards Sci. Paper No. 317. Na, Cd, K, Zn, Tate and Foote, Phil. Mag., 36, p. 64; 1918. Mg, Tl, Foote and Mohler, Phil. Mag. (in press).

A trap containing cadmium-tin alloy chips immersed in liquid air was used between the pumps and the ionization tube in order to preclude the possibility of the presence of mercury in the ionization chamber.

Fig. 2.
Ionization tube used for rubidium and cæsium.

The experimental procedure consisted in applying a constant retarding potential of 0.8 to 1.5 volts between the net and the outside cylinder and measuring both the total current from the hot wire, and that portion of the current which reached the outside cylinder against the retarding field, as functions of a variable accelerating potential applied between the hot wire and net. The apparatus used for rubidium and cæsium is illustrated by Fig. 2. This was constructed of soda glass, as it was found that the heated vapors of these metals do not noticeably attack soda glass while the reaction with lead glass is very pronounced. The net and cylinder were of nickel, and a platinum hot wire coated with lime served as the cathode. A retarding potential of about 0.5 volt was applied between the net and cylinder. Metallic rubidium from Kahlbaum was used and several grams of cæsium were obtained by heating metallic calcium and Kahlbaum cæsium chloride to about 400° C. at a very low pressure. The metal was distilled into a small test-tube, covered with a drop of benzol, and inserted into the pump intake, after which the apparatus was sealed and evacuated. The metal was then redistilled into the ionization chamber. During the observations the

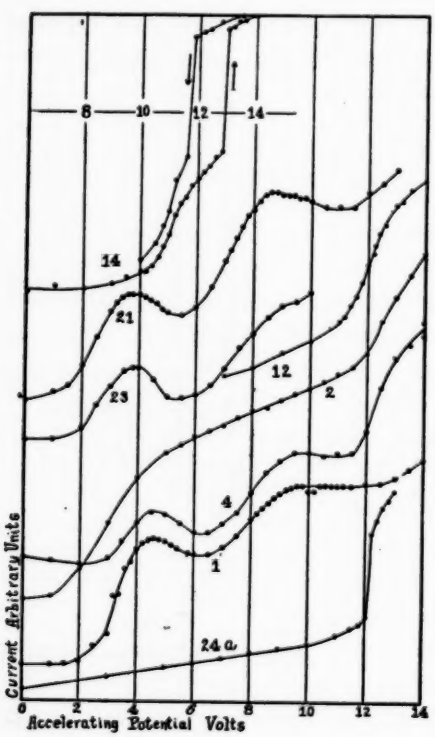


Fig. 3.
Total and partial current curves for arsenic.

entire ionization tube was uniformly heated to about 150° C., and the metal gradually distilled out, condensing in the exhaust tube. When necessary the metal was redistilled back into the ionization tube.

TABLE I.

Ionization and Resonance Potentials for Electrons in Arsenic Vapor. [Curves 1 to 12; Cadmium-tin alloy trap and liquid-air trap between pump and ionization tube for several hours before and during the time of observation. Curves 21 to 24a: All apparatus reconstructed and both traps used continuously from the time the apparatus was assembled.]

Curve.	Resonance.			Applied Potential at Ionization.	Initial Potential.	Ionization Potential.[1]	Resonance Potential.[2]	
	a.	*b.*	*c.*				*b–a.*	*c–b.*
1	3.6	8.4	...	...	..	...	4.8	..
2	..	..	...	10.4	1.0	11.4	..	..
3	..	..	...	10.5	1.0	11.5	..	..
4	3.8	8.4	12.6	...	..	...	4.6	4.2
4*a*	3.8	8.4	12.8	...	..	...	4.6	4.4
5	..	..	...	10.6	1.0	11.6	..	..
10	3.6	8.8	...	...	..	...	5.2	..
12	..	..	...	10.6	1.1	11.7	..	..
21	2.8	7.7	...	...	..	...	4.9	..
22	..	..	...	10.0	1.8	11.8	..	..
23	3.0	7.8	...	...	..	...	4.8	..
23*a*	3.2	..	...	...	..	...	..	..
24	..	..	...	9.6	1.6	11.2	..	..
24*a*	..	..	...	10.0	1.6	11.6	..	..

Fig. 3 represents some of the data obtained for arsenic. A complete analysis of these and other curves is given in Table I. The partial-current curves show pronounced changes in curvature at successive points *a*, *b*, *c*, which differ in potential by a constant amount. This constant difference gives the resonance potential directly, eliminating any consideration of initial velocities. The total-current curves show a rapid increase in slope at a point for which the potential, corrected for the initial velocity (potential) of the electrons as obtained from the partial-current curves, gives the ionization potential. The initial potential is the difference between the resonance potential and the applied potential corresponding to the first point of resonance. A detailed discussion of this method for determining the ionization potential has been given elsewhere.[3] Curve 14, Fig. 3, is of interest as an example of a discontinuous total-current curve. On increasing the accelerating potential the total current increases gradually up to about 13 volts. At this point the current increases discontinuously and upon decreasing the

[1] Mean ionization potential = 11.5 volts.

[2] Mean resonance potential = 4.7 volts.

[3] Tate and Foote, J. Wash. Acad. Sci., 7, p. 520, 1917.

accelerating potential the current decreases discontinuously at about 12 volts. Similar curves have been obtained for mercury, sodium and rubidium. Curves of this type may be secured when the vapor density and the electronic current are high. In the present instance a measurement of the initial velocity of the electrons was not made but, in general, these discontinuous curves afford as accurate a determination of the ionization potential as those obtained with a smaller vapor density and initial velocity. The discontinuous change occurs considerably above the ionization potential and is probably due to a potential redistribution in the ionization chamber. As the potential is gradually increased the most intense part of the discharge, at least for rubidium, sodium, etc., for which metals the glow discharge is visible, moves suddenly at the point of discontinuity from the hot wire to the net.

The final results with arsenic gave 4.7 volts for the resonance potential and 11.5 volts for the ionization potential. These values are quite similar to the corresponding voltages for mercury, namely, 4.9 volts and 10.3 volts, and for this reason it was at first suspected that mercury from the pumps and gage was present in the ionization chamber. Although both liquid air and cadmium-tin alloy were used as a mercury trap in the first series of experiments here described (curves 1 to 12) the apparatus was entirely reconstructed with every precaution again being taken to avoid contamination by mercury. The results of this second series of experiments (curves 21 to 24*a*) confirmed those of the first series. In work at this laboratory with eleven different metals no trace of mercury has ever been observed, when the metals are volatilizing rapidly, in the characteristic ionization and resonance curves, although heretofore we have never used either cadmium-tin or liquid-air traps. In the case of the gases and non-metallic vapors, however, considerable precaution must be taken to avoid the presence of mercury. It was therefore thought possible that contamination by mercury might still be present and show up prominently in the somewhat non-metallic arsenic vapor. The following experiment was accordingly performed.

It was noted that, contrary to our experience with other metals, no glow discharge could be detected in arsenic at the ionization point. A special discharge tube was constructed for observing the arsenic arc more carefully. A steel disk anode and a tungsten hot-wire cathode were mounted about 1 cm. apart in a glass tube lined with iron net in electrical contact with the hot wire. The tube was mounted in a special electric furnace having a small side opening through which the arc could be viewed. Above the distillation point of arsenic the electron current indicated marked ionization as before at the ionization potential

of 11.5 volts. With a current of 30 milliamperes no arc or glow discharge whatever was visible. A few drops of mercury were then placed in the ionization tube. A visible arc was obtained with a current of 0.02 milliampere, and with a current of 20 milliamperes the arc was very intense. Thus the fact that a current of 0.02 milliampere is sufficient to show the presence of mercury as an impurity of arsenic, and that a current of 30 milliamperes in supposedly pure arsenic vapor does not show a trace of any glow whatever, must be taken as a proof of the purity of the vapor. It is of further interest to note that when arsenic is heated with a small amount of mercury the ionization due to the mercury will appear 200° C. below the point where arsenic vaporizes sufficiently to show ionization, and after prolonged heating at 500° C. the mercury is driven out of the tube so that finally the ionization of arsenic alone is observed and the glow discharge disappears.

It has been shown in the earlier papers referred to that for many metals the ionization and resonance potentials are determined by the quantum relation $h\nu = eV$ where h is Planck's constant of action, e the electronic charge, V the resonance or ionization potential, and ν a frequency determinable from spectroscopic data. The value assigned to ν appears to depend upon the column of the periodic table to which the particular metal under consideration belongs. Thus, for Group I. of the periodic table, ν is the frequency of the first line of the principal doublet series for resonance potential and the convergence of this series for ionization potential. For Group II. the resonance potential is determined by the first line of the combination series of single lines 1.5 S–mp_2, and the ionization potential is determined by the convergence of this series. Arsenic is the first metal in Group V. so far investigated. The quantum relation may be written $\lambda = 12334/V$ on substituting the proper values for h and e, where λ is the wave-length in Ångstrom units corresponding to the potential V expressed in volts. Thus, the resonance potential of 4.7 volts and the ionization potential of 11.5 volts correspond to the wave-lengths $\lambda = 2620$ Å. and $\lambda = 1070$ Å., respectively. No series having its first line at $\lambda = 2620$ Å. or its convergence at $\lambda = 1070$ Å. is known for arsenic. In fact, all the series relations so far observed in arsenic have been quite unsatisfactory, which is true also of the other metals in this family. Accordingly, the present work very likely indicates an undiscovered series in arsenic having its first term (or terms if a doublet series) near $\lambda = 2620$ Å. and converging near $\lambda = 1070$ Å., but as to the character of the series nothing can be inferred.

Fig. 4 represents the partial-current curves and Fig. 5 the total-current curves obtained with rubidium. As seen from Table II., the mean

value of the resonance potential is 1.6 volts. This value and the first point of resonance collision, a, permit the determination of the initial velocities or equivalent potentials shown in column 9. The total-current curves 1, 6, 11, 12 were taken under the same experimental conditions as the partial-current curves 3, 4, 5, so that the initial potential, V_0, to be added to the observed applied potential at ionization to give the ionization potential, is 1.2 volts. Curve 12 corresponds to curve 7 with an initial potential of 1.4 volts, the cathode being operated

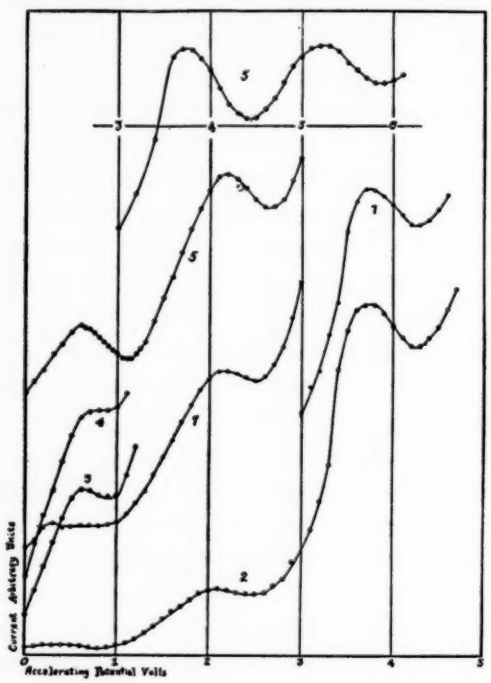


Fig. 4.

Partial-current curves for rubidium.

at a higher temperature when these two curves were obtained. In the case of rubidium and cæsium the total current did not show saturation below the ionization potential, so that the point at which ionization begins is not indicated by as sharp a break as would otherwise occur. The following method for determining the break point proved quite satisfactory.

Langmuir[1] has derived an equation showing the relation between the thermionic current and voltage for the case of a hot wire surrounded by a cylindrical anode. If account is taken of the initial velocity of the

[1] PHYS. REV., 2, p. 457, 1913.

electrons, an additional constant appears in the equation derived by Langmuir. The kinetic energy of an electron at a distance r from the hot wire is:

$$\frac{1}{2}mv^2 = (V + V_0)e, \tag{1}$$

where V_0 is the initial potential and $V = f(r)$ is the applied accelerating potential. Defining i as the thermionic current per unit length of the hot wire, one obtains directly from Poisson's equation:

$$r\frac{d^2V}{dr^2} + \frac{dV}{dr} = i\sqrt{\frac{2m}{e(V + V_0)}}, \tag{2}$$

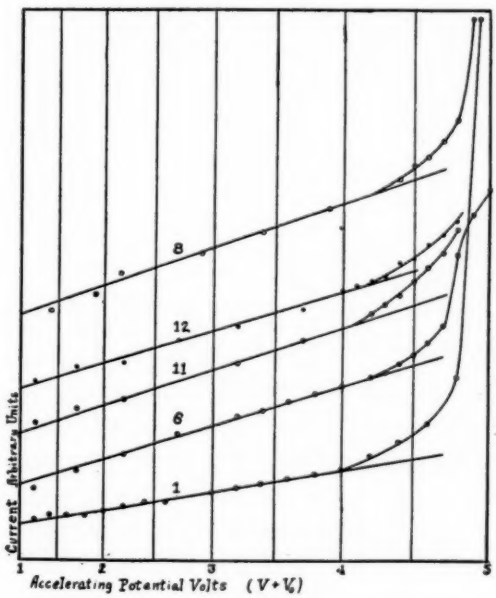


Fig. 5.

Total-current curves for rubidium.

the solution of which may be represented by the following relation where A is a constant for any definite configuration of the ionization chamber and temperature of the hot wire.

$$i = A(V + V_0)^{3/2}. \tag{3}$$

Hence, the relation between i and $(V + V_0)^{3/2}$ is linear for the purely thermionic current below its saturation value. When ionization begins the current increases more rapidly so that the ionization potential is given by the value of $(V + V_0)$ at the point where the curve i plotted against $(V + V_0)^{3/2}$ deviates from a straight line. Fig. 5 represents the

total-current curves for rubidium plotted in this manner. The values of the applied potential were corrected by adding the initial potential V_0 as given in Table II. The linear relation is well satisfied except possibly at the lowest potentials. The agreement is even better than

TABLE II.

Ionization and Resonance Potentials for Electrons in Rubidium Vapor.

Curve.	Resonance.				Resonance Potential.[1]			Initial Potential.
	a.	*b.*	*c.*	*d.*	*b–a.*	*c–b.*	*d–c.*	
2.....	..	1.65	3.35	...	...	1.70	...	
3.....	0.4	...	...	...	...	...	...	1.2
4.....	.4	...	...	...	...	...	...	
5.....	.4	1.85	3.45	4.90	1.45	1.60	1.45	
7.....	15	1.70	3.50	...	1.55	1.80	...	1.4
					Mean = 1.6 volts.			
	V_0.	$V+V_0$[2].						
1.....	1.2	4.0						
6.....	1.2	4.2						
8.....	1.4	4.2						
11.....	1.2	4.1						
12.....	1.2	4.2						

might be expected from a more careful consideration of the applicability of equation (3). Strictly considered, this equation should not hold exactly for two reasons: (1) Because of inelastic collision of some of the electrons at the resonance potential, and (2) because of the potential drop along the hot wire. Thus, equation (3) states that when $V = 0$, $i = A V_0^{3/2}$, but, although V_0 is comparable with V, the observed current is practically zero when the applied field is zero. The reason for this is that the electrons with no accelerating field between the hot wire and net fall along the hot wire from points of lower to higher potential or along one of the leading-in wires. A measurable current is observed only when the applied potential is comparable with the electrostatic field along the wire. This fact accounts in part for the high initial velocities observed. Also it may be noted that our method of selecting the point of resonance gives greatest weight to the electrons

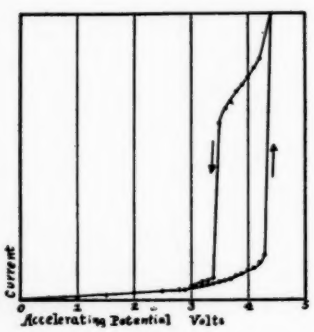


Fig. 6.
"Discontinuous" total-current curve for rubidium.

[1] Resonance potential = 1.6 volts.
[2] Ionization potential = 4.1 volts.

of highest speed. A point of further interest is that practically the same values for the initial velocity are obtained when the partial-current curves are plotted directly and when they are plotted on the rectified coördinate system used for the total-current curves.

The mean value of the ionization potential of rubidium obtained from the curves shown in Fig. 5 is 4.1 volts. Fig. 6 illustrates a type of discontinuous total-current curve obtained when the density of the rubidium vapor is high.

TABLE III.

Ionization and Resonance Potentials for Electrons in Cæsium Vapor.

Curve.	Resonance.		Resonance Potential.[1]	Initial Potential.
	a.	*b.*		
3	0.60	2.10	1.50	0.88
5	.55	2.00	1.45	.93
7	.50	2.00	1.50	.98
8	.55	2.00	1.45	.93
10	.55	2.00	1.45	.93
11	.55	2.00	1.45	.93
12	.40	1.90	1.50	1.08
14	.40	1.90	1.50	1.08
Mean			1.48	1.0
	$(V+1.0)^{\frac{2}{3}}$.	Ionization Potential.[2]		
1	8.5	4.2		
4	7.7	3.9		
6	7.6	3.9		
9	7.1	3.7		
13	7.4	3.8		

Fig. 7 shows the partial-current curves and Fig. 8 the total-current curves for cæsium vapor. The method of determining the ionization and resonance potentials was that described above for rubidium. The final values, summarized in Table III., are 1.48 volts for the resonance potential and 3.9 volts for the ionization potential.

From the behavior of sodium and potassium one would expect the resonance and ionization potentials of rubidium and cæsium to be determined by the relation $V = 12334/\lambda$ where λ is the first line and convergence wave-length, respectively, of the principal series of doublets for each metal. The following table, to which for completeness have been added the data on the other alkali metals, confirms this fact.

[1] Resonance potential = 1.48 volts.

[2] Ionization potential = 3.9 volts.

TABLE IV.

Resonance and Ionization Potentials for the Alkali Metals.

Metal.	λ (Å). ($\nu = 1.5^3 - 2p$).	Resonance Potential Volts.		λ (Å). ($\nu = 1.5\ s$).	Ionization Potential Volts.	
		Computed.	Observed.		Computed.	Observed.
Lithium....	6707.85[1]	1.839	....	2299.67[2]	5.363	....
Sodium.....	5895.94[1]	2.092	2.12[3]	2412.63[4]	5.112	5.13[3]
	5889.97[1]	*2.094*				
Potassium..	7699.01[1]	1.602	1.55[3]	2856.69[5]	4.318	4.1[3]
	7664.94[1]	*1.609*				
Rubidium ..	7947.64[1]	1.552	1.6[6]	2968.40[5]	4.155	4.1[6]
	7800.29[1]	*1.581*				
Cæsium....	8943.46[1]	1.379	1.48[6]	3184.28[5]	3.873	3.9[6]
	8521.12[1]	*1.447*				

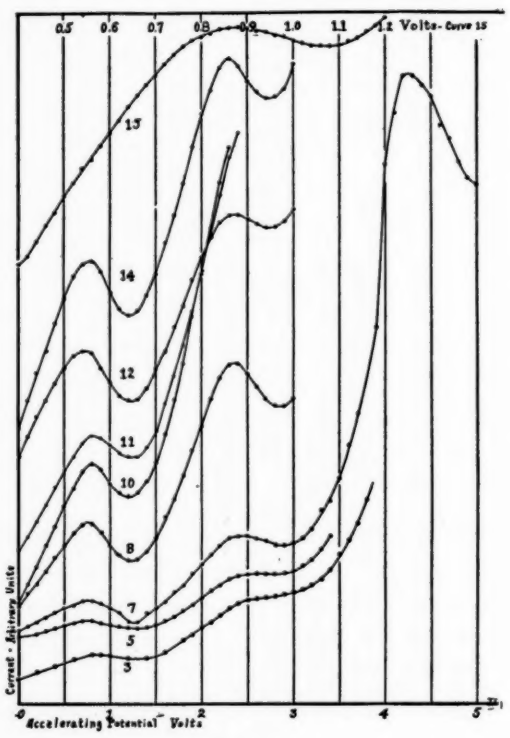


Fig. 7.

Partial-current curves for cæsium.

[1] Meggers, Bur. Standards Sci. Paper No. 312.
[2] Dunz Tübingen Thesis, 1911.
[3] Tate and Foote, Phil. Mag., 36, p. 75, 1918.
[4] Wood and Fortrat, Astrophys. Jour., 43, p. 73, 1916.
[5] Watts, Phil. Mag., 29, p. 775, 1915.
[6] Foote, Rognley, and Mohler, this paper.

The first pair of doublets in the principal series of both rubidium and cæsium is rather widely separated. In the case of cæsium the separation is equivalent to about 0.07 volt. If resonance collision occurs at both 1.379 and 1.447 volts, corresponding to each member of the cæsium doublet, the partial-current curves should show two sets of maxima differing by 0.07 volt at the first point of resonance, by 0.14 volt at the second point of resonance, etc. None of the curves of Fig. 7 show double

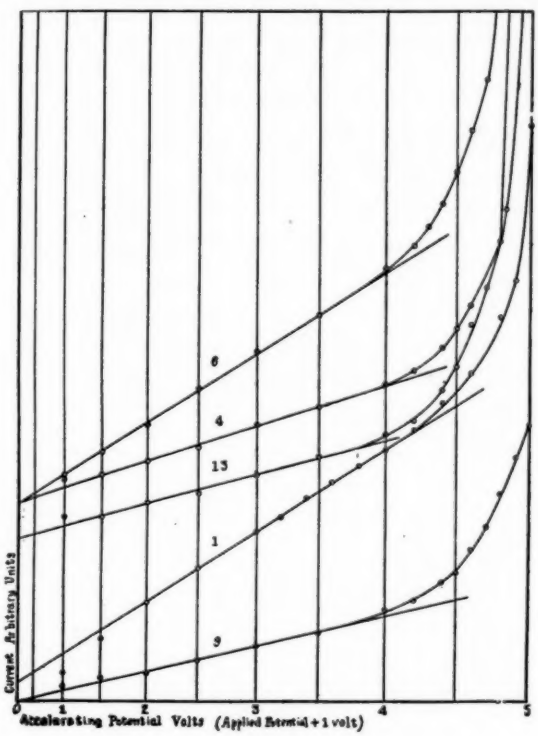


Fig. 8.

Total-current curves for cæsium.

maxima and for curve 15, observations were made at every 0.02 volt. It is possible that the presence of double maxima could not be detected on account of the initial velocity distribution from the hot wire. However, it has been found by Foote and Mohler[1] that in the case of thallium the shorter wave-length member of the first doublet of the principal series appears to determine the resonance potential. Hence it seems quite likely that the proper theoretical values for the resonance potentials of rubidium and cæsium are 1.581 and 1.447 volts, respectively. This possibility suggests an analogy to the general behavior of the *K*

[1] PHYS. REV., 11, p. 487, 1918.

series in the X-ray spectrum of metals as has been pointed out by Foote and Mohler.[1] The present work indicates that the "single line" spectra of rubidium and cæsium are the doublets $\lambda = 7949$ and 7803 Å. and $\lambda = 8950$ and 8528 Å., respectively. At the ionization potential the principal series, the first subordinate series, and the second subordinate series appear. The following doublets were observed visually in the case of cæsium when the accelerating potential was equal to the ionization potential: Principal series, $m = 3$; first subordinate series, $m = 5, 6, 7, 8, 9, 10$; second subordinate series, $m = 4.5$.

SUMMARY.

Two types of inelastic collision between the electron and atom occur in vapors of arsenic, rubidium and cæsium. The potential differences through which an electron must fall to attain sufficient energy or velocity to produce these collisions are known as the resonance and ionization potentials for the particular metal in question. For rubidium and cæsium the resonance potential is determined by the quantum relation $h\nu = eV$ where ν denotes the frequency corresponding to the first doublet of the principal series. A similar relation holds for the ionization potential where ν denotes the convergence frequency of the principal series. Experimentally determined values of the resonance and ionization potentials for rubidium were 1.6 volts and 4.1 volts, respectively; for cæsium, 1.48 volts and 3.9 volts, respectively; for arsenic, 4.7 volts and 11.5 volts, respectively. By applying the quantum relation to the results obtained with arsenic we are able to predict the presence of an undiscovered spectral series in arsenic having its first term (or terms if a doublet series) near $\lambda = 2620$ Å. and converging near $\lambda = 1070$ Å. A Wehnelt tube discharge in arsenic possesses no luminosity indicating that the important series lines lie in the ultra-violet. At the ionization potential of rubidium and cæsium the principal, first, and second subordinate series appear. No ionization whatever could be detected at the resonance potentials of rubidium, cæsium, and arsenic.

BUREAU OF STANDARDS,
WASHINGTON, D. C.,
August 16, 1918.

[1] J. Wash. Acad. Sci., Sept. 19, 1918.

ENERGY OF THE CHARACTERISTIC X-RAY EMISSION FROM MOLYBDENUM AND PALLADIUM AS A FUNCTION OF APPLIED VOLTAGE.

By Benjamin Allen Wooten.

Introduction.

SINCE the introduction of the X-ray spectrometer by W. H. and W. L. Bragg as an instrument for studying the various properties of X-radiation, much work has been done. Most of this work concerns itself with the general radiation and there is relatively little published matter on the intensity of the characteristic rays. In a recently published article[1] D. L. Webster finds that the rays of the K series of rhodium are produced only at voltages above that required by the Planck quantum hypothesis to excite general radiation of a frequency slightly greater than that of the γ line. He finds that above that potential all K lines increase in the same ratio for a given increase in potential. The rates of increase of intensity are continuously accelerated.

The following work was undertaken with a view of studying the relation between the energy of the lines of the K series of molybdenum and palladium and the voltage producing them. The spectrometer that was used was similar in general to the one used by Dr. Clayton T. Ulrey[2] in his work on the general radiation. Voltages were measured by means of an electrostatic voltmeter and energy was supplied for the X-ray tubes by a Kenotron rectifying set. During the progress of the work points concerning the probability of new lines, absorption coefficients, etc., presented themselves. They were dealt with and will be discussed.

Description of Apparatus.

Current at 125 volts and 500 cycles was taken from a motor generator set and was stepped up by a three-kilowatt oil-immersed transformer. The high voltage alternating current from the transformer was rectified by means of two General Electric Kenotrons and an oil-immersed glass-plate condenser. The energy supplied to the X-ray tube was taken from this condenser. Under normal conditions the voltage generated by the

[1] D. L. Webster, Phys. Rev., June, 1916, pp. 599–613.

[2] C. T. Ulrey, Phys. Rev., May, 1918.

set was constant to one and one half per cent. The voltage was measured by an electrostatic voltmeter that was built in the laboratory. It consisted of a pair of fixed spheres and a pair of movable ones hung on a steel strip suspension and pivoted at the bottom on a needle. The four spheres were charged to the same potential, and the force of repulsion between the fixed and movable ones caused the instrument to show a deflection.

The voltmeter was calibrated by means of a standard spark gap. This gap consisted of 12.5 cm. balls. The relation between sparking distance and voltage was obtained from published tables. The calibration curve of the voltmeter was very regular, almost approaching a straight line at high voltages. The voltmeter kept its adjustment perfectly.

Currents to the X-ray tube were measured on a Weston direct current milliammeter in all the work except that on absorption. In that part of the work the currents were read on a Leeds and Northrup portable galvanometer properly shunted. An accident to the milliammeter made this change necessary.

The X-ray tube that was used in the work on molybdenum was one that was kindly supplied by Dr. W. D. Coolidge, of the General Electric Company. The Palladium tube was built up in the laboratory. The cathode was of the Coolidge type and it was also furnished by Dr. Coolidge. The anode consisted of a cylinder of iron with a small piece of palladium set in it. The target face was inclined at an angle of 26 degrees with the plane perpendicular to the elements of the cylinder. The tube was evacuated by means of a diffusion pump of the Langmuir type, the mercury trap of which was cooled with a mixture of carbon dioxide snow and ether. This pump was in series with a Gaede rotary pump and a piston pump. The tube was baked at 300° Centigrade while the preliminary evacuation was in progress and the target was heated to dull redness from time to time by discharges at 25 kilo-volts. It was found necessary to keep the pump in operation whenever observations were being made.

The spectrometer was peculiar only in the fact that the ionization chamber was very long. Its length was 128 cm., and complete absorption through a long range of wave-lengths was possible with its use. This chamber contained the saturated vapor of methyl iodide. Methyl iodide was used because it does not show abnormal absorption at any point throughout the range of wave-lengths investigated.[1] The chamber and electrometer were freed of water vapor by the use of phosphorous anhydride. The gold-leaf electrometer was attached directly to the end

[1] "X-rays," G. W. C. Kaye, p. 147.

of the ionization chamber, and the image of the leaf was projected on a ground glass screen with a scale marked on it. This screen was securely fastened to the ionization chamber. The leaf was charged negatively to 400 volts by means of a small dynamotor. The X-ray tube was surrounded by a heavy lead screen and a beam was taken off at about 26 degrees from the face of the target, this beam coming as nearly as possible from the center of the focal spot. After passing through an aperture in the screen this beam was passed through an adjustable slit, the jaws of which were of lead, and then reflected by the crystal. All circuits to the tubes, voltmeter, etc., carrying high potential current returned through ground. The Coolidge cathodes were heated by current from a small storage battery.

The crystal used was of Montana calcite and was kindly lent to us by Professor A. J. Moses, of the Department of Mineralogy. Photographs of the back slit taken with the reflected beam showed remarkably clear and perfect lines, and it was not possible to detect any irregularities in them.

The natural leak of the electrometer when properly connected to the ionization chamber was such as to cause the leaf to move across the interval of the scale that was used in taking readings in about 625 seconds. This of course varied, but it was usually more than 600 seconds. The readings of X-ray intensity were taken in the usual way by allowing the reflected beam to pass into the chamber through an adjustable slit, and noting with a stop watch the time taken for the leaf to pass across a fixed part of the scale. The same scale interval was used in all the readings taken. The time taken for a reading varied from less than one second to 550 seconds. The projected image of the leaf on the ground glass screen was clear and sharp and readings could be repeated with an accuracy of one per cent.

Results.

The apparatus was adjusted and lined up so that a beam of X-rays from the back slit would pass over the center of the spectrometer circle. The position of the ionization chamber, when this beam was entering its slit was read on the circle and taken as its position for direct rays.

The crystal was then put in place and adjusted so that its reflecting face was as nearly as possible parallel to the beam of direct rays and over the axis of the spectrometer. The crystal was moved in steps of θ and the ionization chamber in steps of 2θ until strong reflection was observed. The crystal was then held stationary while the chamber was moved in small steps across the reflected beam. From the readings of intensity

obtained for each step and the position of the chamber at each step a curve showing a sharp peak at the position of the chamber for maximum intensity was plotted. The position of the chamber corresponding to this peak was taken as its position when the reflected beam was passing correctly through its slit.

In obtaining the values of intensity from the readings of the rate of fall of the leaf, this rate was first corrected for natural leak and then reduced to what it would have been for a current of one milli-ampere in the tube.

As the nature of the work undertaken was not such as to demand extreme accuracy in wave-length measurements, the angular readings were not corrected for depth of penetration of the beam into the crystal. The natural leak was always taken when the apparatus was in operation with the chamber well out of line. The walls of the ionization chamber were thick enough to shield out all the secondary radiation at the highest voltages that were used in this work, and no correction for it was necessary.

VOLTAGE INTENSITY CURVES.

After properly lining up the apparatus data was taken on intensity for different wave-lengths at constant voltages up to and including 50 KV. The readings were taken at intervals of three minutes of arc and the intensities were corrected for natural leak and reduced to values for unit current. Curves were taken for different voltages and their general shapes noted for both molybdenum and palladium. Fig. 1 is such a curve taken for palladium and it is shown as being typical of the ones obtained. These curves were taken in order to locate the more prominent lines in the K series and to show the general nature of the spectra of the metals.

In order to study the intensity of the characteristic radiation it was necessary to subtract the intensity of the general radiation of the same wave-length as a given line from the intensity of the line as measured by the spectrometer. With a number of the constant voltage, intensity and wave-length curves at hand, a point in the general radiation on each side of a line (α or β) was taken. These points were chosen so that a straight line through them would pass through the line at the same place that a continuation of the curve of general radiation would cut it. It was found necessary to take a great many curves in order to choose these points correctly.

Having chosen these points for the α and β line of each of the metals, data was taken in the following manner: The spectrometer was set for

the wave-length of one of the chosen points, care being taken to have the setting correct. In order to make certain of this, the crystal was set on the proper reading as determined from the wave-length intensity curves and the ionization chamber moved past the reflected beam in steps as described above. The results of reflected intensity were plotted against angular displacement of the chamber and the chamber was set at the reading corresponding to the peak of this intensity curve. In all settings

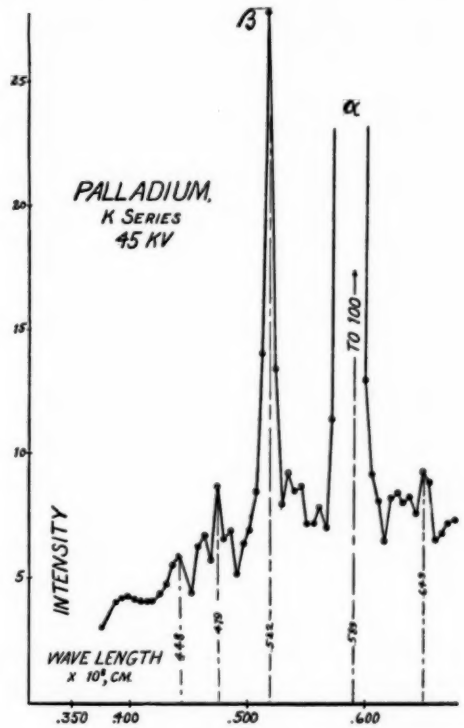


Fig. 1.

made in this part of the work this procedure was gone through. The voltage on the tube was increased in steps to 50 KV., and with each change of potential the current through the tube was adjusted so as to keep the energy supplied to the tube constant. Determinations were made of intensity of reflection for each value of voltage and all intensities were reduced to unit current conditions. The spectrometer was then set for the wave-length of the other point and similar readings taken for it, giving the same values to potential and current that were used before, for each step. Then the spectrometer was set for the line under consideration and values for its intensity for the voltages and currents that

were used in the determinations on the two wave-lengths in the general radiation were obtained. This data was determined for each line that was studied.

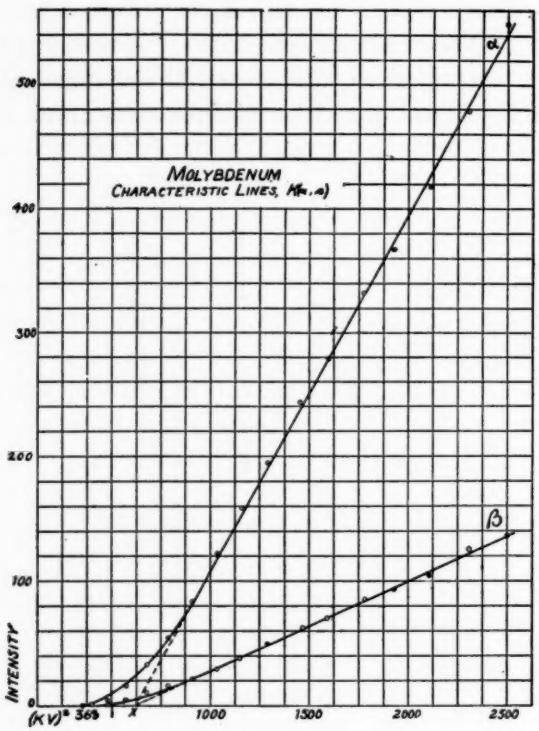


Fig. 2.

The three intensities for any potential, determined in this way give three points which determine a triangle with its acute angle very small. The distance between the vertex of this acute angle and the base line, as measured along the bisector of the angle, gives the intensity of the radiation of the line corrected for the general radiation of the same wave-length. The shape of the triangle thus formed is such as to make the length of this bisector sensibly equal to the length of either one of its sides except for low voltages.

It was found that the α and β lines for each metal appeared together at a certain minimum voltage. In the case of molybdenum this voltage was found to be 19.2 KV. and for palladium it was 24 KV. Below these potentials the wave-length curves were smooth and they intersected the wave-length axis at points corresponding to longer wave-lengths than the wave-length of the β line. It was shown by Duane and Hunt[1] that

[1] Duane and Hunt, American Physical Society Proc., April, 1915, PHYS. REV., August, 1915.

this point of intersection of the curve with the wave-length axis represented the minimum wave-length that the potential at which the curve was taken could give by the quantum relation. This was found to be true for these curves.

Curves were plotted between the kilo-volts squared to the tube and the intensity obtained as above of the lines for each metal. These curves are shown in Figs. 2 and 3 for molybdenum and palladium re-

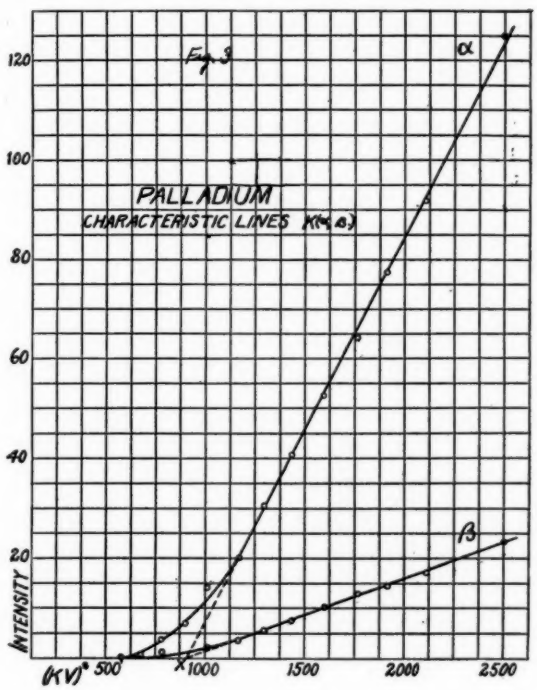


Fig. 3.

spectively. It will be noticed that the two sets of curves, each set consisting of a curve for the α and one for the β line of a metal, are of the same general shape and character. In each set the α and β curves rise from a common point on the voltage squared axis corresponding to the square of the lowest potential that will excite the line or the series, and quickly become straight lines. The intensity of the radiation in the α and β lines for the two metals thus appears to vary as the square of the voltage applied for voltages not too near the critical voltage; and the ratio of the intensity of the α to the intensity of the β line in each case quickly becomes constant for voltage variations.

Below is a table (Table I.) of the results of various measurements

taken in determining these curves, together with other data concerning them.

TABLE I.

	Molybdenum.	Palladium.
Atomic number	42	46
Wave-length of α line (cm. $\times 10^8$)	.716	.589
Wave-length of β line (cm. $\times 10^8$)	.641	.522
Minimum potential to produce K series, in kilo-volts	19.2	24.0
Point of intersection of straight line parts of curves with (voltage)^2 axis, (kilo-volts)^2	625	930
Ratio of intensity of α to β line as measured from data	3.99	5.28
Ratio of intensity of α to β line corrected for absorption in glass	5.55	6.25

ABSORPTION COEFFICIENTS.

A theoretical explanation of these voltage intensity curves has been developed by Professor Bergen Davis[1] so it was decided to correct the curves for absorption in the glass of the X-ray tubes and to compare the experimentally determined results with the curves obtained from his equation. An investigation of the absorption coefficients of the α and β lines of molybdenum and palladium in glass and in molybdenum and palladium respectively was undertaken.

The radiation was taken from a Coolidge tube with a tungsten anticathode and the wave-lengths desired were reflected from the same crystal that was used in the other part of the work. The voltage on the tube was made greater than the critical voltage of either metal for the K series and it was held constant. The collimator slit and the slit on the ionization chamber were narrowed to .2 mm. and the crystal and chamber were set at the proper angles to give the desired wave-length. An aluminum screen .3 mm. thick was placed between the crystal and the slit on the ionization chamber so that the reflected beam would pass through it. It was found necessary to use this screen in order to get consistent results. It absorbed practically all of the secondary radiation from the crystal, and thus allowed the reflected ray to pass into the ionization chamber as an homogeneous beam. This filtering out of the X-rays of long wavelength that came from the crystal was important.

The reflected beam after having passed through the aluminum filter was passed directly into the ionization chamber and its intensity determined. This value was used as the initial intensity in the computations. Then thin sheets of glass were placed in front of the slit in such a way that

[1] Bergen Davis, PHYS. REV., June, 1918.

the reflected beam passed through them before it entered the ionization chamber. Determinations of intensity were made with the rays passing through various thicknesses of glass and the absorption coefficient calculated by the well-known exponential equation. The results obtained with different thicknesses of glass were averaged for the true absorption coefficient.

Measurements were made in this way for the α and β lines of both molybdenum and palladium in glass. In addition absorption coefficients of the α and β line of molybdenum in molybdenum were determined, as were the absorption coefficients of the α and β lines of palladium in palladium.

The aluminum filter was kept in place in all this work and conditions were the same as regards voltage, current, etc., in all the determinations that were made. In one set of observations the potential across the tube was 35 KV. and in another it was held at 45 KV. The results shown in this paper were those obtained at 45 KV. The sheets of glass that were used were .022 cm. in thickness and the sheets of palladium were .0065 cm. thick. The molybdenum was .016 cm. thick and only two thicknesses were used. The absorption coefficients obtained are given in Table II.

TABLE II.

Line.	Wave-length Cm. $\times 10^8$.	Absorption Coeff. in Glass.	Absorption Coeff. Mo.	Absorption Coeff. Pd.
Mo. α.....	.716	14.50	210	
Mo. β.....	.641	11.65	170	
Pd. α.....	.589	7.79		138
Pd. β.....	.522	5.37		100

WAVE-LENGTH INTENSITY CURVES.

During the early part of the work a great many curves were obtained between wave-length and intensity of radiation for molybdenum. The readings were taken for intervals of three minutes of arc on the spectrometer circle and in no case was the α line divided. The α line as obtained is wider than the β, and this widening at the base suggests that the α line consists of two lines. It has been shown by Professor Duane[1] that the α line of rhodium is a doublet consisting of an α_1 and an α_2 line. In this work no distinction is made between the α_1 and the α_2 lines and the radiation designated here as that of the α line for either metal consists of the combined intensities of the two lines of the α doublet.

In order to correct the intensities of the lines that were studied (α and

[1] Not yet published.

β lines) for absorption in the glass of the X-ray tubes, the thickness of the glass was determined. A tube was placed under a microscope and the microscope was focused on an object on the inner surface of the glass and then on an object on the outer surface. From these readings and the index of refraction of the glass the thickness of the glass was determined. Determinations were made of the thickness of both tubes used and the results were:

Thickness of glass of molybdenum tube.....1125 cm.

Thickness of glass of palladium tube.....0706 cm.

Using these thicknesses and the absorption coefficients in Table II., the intensities within the tubes of the lines studied were found to be

Moα .....5.11 times the observed intensity.
Moβ .....3.67 " " " "
Pdα .....1.73 " " " "
Pdβ .....1.46 " " " "

The curves between intensity of radiation and voltage squared (Figs. 2 and 3) show that for the straight line parts of the curves the ratio of the intensity of the α radiation to that of the β radiation is, in the case of molybdenum, 3.99; and in the case of palladium 5.28. When the intensities of the lines are corrected for absorption in the glass these ratios are changed for molybdenum to 5.55 and for palladium to 6.25. The curves in Figs. 2 and 3 are the experimentally determined curves and they are not corrected for this absorption.

THE RATIO OF THE ENERGY IN THE α AND β RADIATION AS EMITTED FROM THE ATOM.

The corrected values give the intensities of the α and β radiation inside the tube before coming out through the glass. *A still more important matter is the ratio of the intensities of these frequencies actually emitted by by the atom.* The theory developed by Professor Bergen Davis[1] makes the calculation of this ratio possible. The equation resulting from a mathematical treatment of this theory is

$$I_a = E_a (hn_a)\frac{BN}{b}\left[\frac{b}{cu}\left(1 - e^{-\frac{cu}{b}(V^2 - V_0^2)}\right)\right.$$

$$\left. - 2V_0(V - V_0) + 2/3\left(\frac{cu}{b}\right)V_0\{2V^3 - V_0(V^2 - V_0^2)\}\right]$$

[1] *Loc. cit.*

$$- \frac{V_0}{15}\left(\frac{cu}{b}\right)^2\{8(V^5 - V^4V_0) - 4V^2V_0(V^2 - V_0^2)$$

$$- 3V_0(V^2 - V_0^2)^2\}\Bigg]$$

when expanded to four terms.

V_0 = minimum voltage required to produce the characteristic radiation.

u = the absorption coefficient of a line in the metal that produces it.

c = the ratio of the path of a ray through the metal to the depth of penetration of the electron producing it.

b = a constant corresponding to a in Whiddington's law.[1] This constant is expressed for use here in terms of potential.

V = any voltage applied to the tube (Kilo-volts).

I = intensity of the line (as written above it means the intensity of the α line).

N = the number of electrons striking the target per second.

B = the number of impacts of an electron with the atomic nuclei per unit distance.

$(E_a h n_a)$ is proportional to the energy emitted by one atom.

It will be noticed that the constants in the equation, with the exception of $E_a h n_a$ and u have the same values for all the lines from a particular metal. The absorption coefficients were determined experimentally for the α and β lines of both palladium and molybdenum and their values are given in Table II. The constants E_α and E_β are the fractions of the effective impacts that result in producing the corresponding radiations.

The combined constants $E_\alpha h n_\alpha$ and $E_\beta h n_\beta$ are proportional to the energies emitted as radiation of each frequency from one atom. The ratio $E_\alpha h n_\alpha / E_\beta h n_\beta$ is the ratio of the energy of the α frequency to that of the β frequency emitted by a single atom.

The experimentally determined values of I for the α and β lines of molybdenum and palladium, after having been corrected for absorption in the glass of the tubes, were plotted. Values of I for different voltages were computed for the α and β lines of both metals and plotted to the same scale. From these computations and from the experimental results the data in Table III. were obtained.

TABLE III.

Metal.	Observed I_α/I_β.	Inside Tube I_α/I_β.	Ratio of Intensities Emitted by Atom $E_\alpha h n_\alpha / E_\beta h n_\beta$.
Mo.....	3.99	5.55	5.70
Pd.....	5.28	6.25	7.18

[1] Proc. Roy. Soc., Vol. 86, 1912.

Professor Bergen Davis's equation can be written:

$$I_\alpha = (E_\alpha h n_\alpha) f_\alpha(V)$$

and therefore

$$\frac{I_\alpha}{I_\beta} = \frac{E_\alpha h n_\alpha f_\alpha(V)}{E_\beta h n_\beta f_\beta(V)}.$$

The computations made in determining the points for the curves give the ratio $f_\alpha(V)/f_\beta(V)$ so that ratio $E_\alpha h n_\alpha / E_\beta h n_\beta$ can be determined. This ratio, the ratio of the energy emitted by an atom as α radiation to that emitted as β radiation, determined in this way is contained with other data pertaining to the comparison of the curves in Table III.

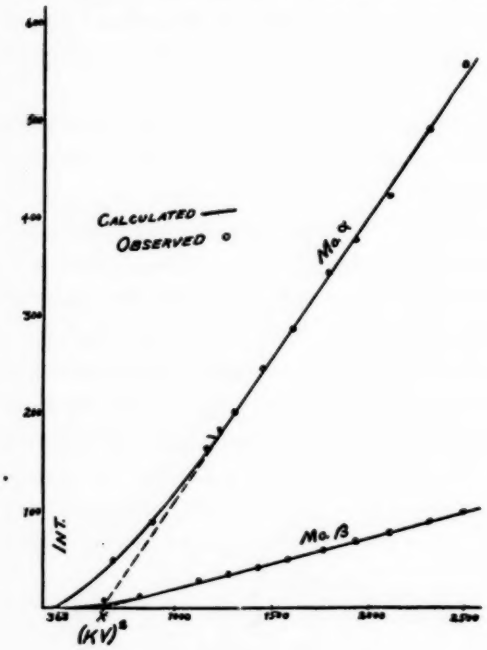


Fig. 4.

The curves given by Professor Bergen Davis's equation plotted to the same scale with the experimentally determined points corrected for absorption, are shown in Figs. 4 and 5. The very close agreement as to shape between the calculated curves and the experimental ones indicates that the assumptions underlying the theory are sound.

In these figures the calculated curves are given in full lines, and the experimentally determined points are marked with circles.

The above comparison is only true in so far as one may consider the reflecting power of the crystal the same for two frequencies not differing

more than the α and β frequencies differ. If the reflecting power of a crystal does depend to a marked degree on the wave-length of the radiation it will be necessary to correct the above ratios for such variation in reflectivity when the same shall be experimentally determined.

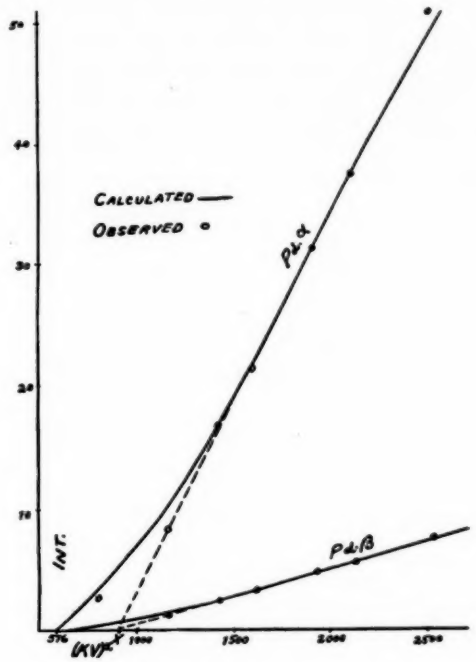


Fig. 5.

Discussion of Results.

In all the work that was done it was assumed that the crystal that was used reflected all wave-lengths with equal efficiency. W. H. Bragg[1] showed that the reflecting power of a crystal for a given wave-length varied with the angle of reflection. He varied the angle of reflection of the beam by taking the different orders of reflection. His results show that the reflecting power of a crystal varies inversely as the square of the order of reflection. This is the only experimental work that has been published as far as the writer knows, and the lack of published matter dealing experimentally with the subject of reflecting power in any single order of different wave-lengths has made the above assumption necessary.

Reference to Figs. 2 and 3 shows that the intensity of the radiation for the α and β lines of the metals used varies directly as the square of the voltage for voltages considerably higher than the critical voltage. In a

[1] W. H. Bragg, Philosophical Magazine, 27, 1914, 881.

paper recently published by Webster and Clark[1] it was shown that the lines of the K series of rhodium vary approximately as the 3/2 power of the difference between the potential applied to the tube and the critical potential of the series. This relation was tried with the data obtained in these experiments but it was found that the curves obtained did not fit the points as well as the ones in Figs. 2 and 3 fit them.

In these curves it will be seen that the linear parts of the curves when produced back cut the voltage squared axis at the same point. In each set of curves this point is marked X. It was found on computation that the following equation is true.

$$\frac{N_{Pd}^2}{N_{Mo}^2} = \frac{V_{0Pd}}{V_{0Mo}} = \frac{\sqrt{X_{Pd}}}{\sqrt{X_{Mo}}},$$

where

N_{Pd} = atomic number of palladium = 46.

N_{Mo} = atomic number of molybdenum = 42.

V_{0Mo} = critical voltage for K series of molybdenum = 19.2 Kv.

V_{0Pd} = critical voltage for K series of palladium = 24.0 Kv.

X_{Mo} = point on axis where straight lines intersect = 625.

X_{Pd} = point on axis where straight lines intersect = 930.

The square root of the value in Kv. squared represented by the point X then, bears the same relation to the metal producing the characteristic radiation that the critical voltage does, *i. e.*, it is proportional to the square of its atomic number.

The method that was used in determining the absorption coefficients is the same as that of the Braggs[2] except for the aluminum filter which was placed at the crystal to absorb its secondary radiation. They find the mass absorption coefficient for the α line of palladium in palladium to be 11.4. The coefficient given in Table I., when divided by the density of palladium gives 12.1. Their work shows that absorption coefficients in a given substance vary as the 5/2 power of the wave-length of the radiation. A. W. Hull and Marion Rice[3] find that absorption coefficients when corrected for scattering vary as the cube of the wave-length, and C. D. Miller[4] finds that they vary for long wave-lengths as the 2.77 power of the wave-length. Miller uses a beam that has passed through a great thickness of absorbing material and in that way absorbs out all but the

[1] D. L. Webster and E. Clark, Nat. Acad. of Science Proc., 3, pp. 181–185, March, 1917.

[2] W. H. and W. L. Bragg, X-rays and Crystal Structure, pp. 175–206.

[3] A. W. Hull and Marion Rice, Proc. Amer. Phys. Soc., PHYS. REV., Sept., 1916, pp. 326–328.

[4] C. D. Miller, PHYS. REV., Oct., 1916, pp. 329–343.

shortest wave-length. The absorption coefficient is determined for this end beam.

The results obtained here, while uncorrected for scattering in the thin sheets of absorbing material that were used, follow the cube of the wave-length law more consistently than they do either of the others. Their agreement with this law is fairly good. The coefficients for the molybdenum α and β radiation in molybdenum are not as trustworthy as the others, because it was not possible to use more than two thicknesses of molybdenum in the screen.

On substituting the observed values of the critical voltage in Planck's equation,

$$Ve = hn$$

in the case of molybdenum, the value of n is such as to correspond to a wave-length of $.643 \times 10^{-8}$ cm. The wave-length of the β line as measured is $.641 \times 10^{-8}$ cm. The measured voltage required to excite the K series characteristic radiation is therefore almost exactly the same as that given by the Planck relation for the β line of the K series of this metal.

For palladium the calculated result, using the observed critical voltage, is a slightly shorter wave-length than that of the β line of the K series of palladium as measured. The measured wave-length of the β line is $.552 \times 10^{-8}$ cm. while the calculated wave-length for the critical voltage is $.516 \times 10^{-8}$ cm.

In a paper published by Uhler[1] it is shown that while Moseley's law does not completely represent the X-ray series over the entire range of wave-lengths, it represents the series for wave-lengths corresponding to the atomic numbers of molybdenum and palladium very closely. Placing $\tau_1 = 2$ in Bohr's equation to determine the wave-length of the α line, the value obtained is, for molybdenum $.713 \times 10^{-8}$ cm., and for palladium $.581 \times 10^{-8}$ cm. The measured wave-length for the molybdenum α radiation is $.716 \times 10^{-8}$ cm. and for palladium α radiation it is $.589 \times 10^{-8}$ cm. In each case the agreement between calculated and observed wave-length is as close as Moseley's law agrees with the observed wave-lengths according to Uhler.

Summary.

The K series characteristic X-radiation of molybdenum and palladium has been studied.

It was found that in each metal the α and β radiation appeared at a certain voltage. For molybdenum this voltage was found to be 19.2 Kv. and for palladium it was 24 Kv. For molybdenum this measured

[1] H. S. Uhler, Phys. Rev., April, 1917, pp. 325–335.

critical voltage is almost exactly that required by the Planck hypothesis to produce the β line. For palladium the measured critical voltage is that required by the same hypothesis to produce a wave-length slightly shorter than the β line.

Curves were plotted between the voltage squared and the intensities of the α and β radiation for both metals, and these curves show that the intensities of these radiations varied directly as the square of the voltage for voltages not too near the critical voltage. The ratio of the intensity of the α line to that of the β line was found to become constant as the voltage was increased, for each metal.

The straight line parts of the curves, if produced back cut the voltage squared axis in the same point, and this point was found to bear the same relation to the atomic number of the metal of the anti-cathode that the point at which the curves themselves cut the axis bears, that is, the square root of the coördinate represents a voltage proportional to the square of the atomic number of the radiator.

Absorption coefficients for wave-lengths of the α and β lines of molybdenum and palladium in glass and in molybdenum and palladium respectively were determined, and they were found to vary approximately as the cube of the wave-length of the radiation. The observed intensities of the α and β lines of each metal were corrected for absorption in the glass of the X-ray tube.

By application of the theory of Professor Bergen Davis it was possible to calculate the ratio of the intensity of the α radiation to that of the β radiation as given by a single atom.

The curves obtained by plotting the theoretical equation were found to agree excellently with the experimentally determined curves.

CONCLUSION.

In conclusion the writer wishes to thank Professor Bergen Davis, who suggested the research, for the active help and constant advice that he has furnished throughout the work. He is deeply indebted to Columbia University for the grant of the Tyndall Fellowship for the prosecution of the research.

He is also grateful to Dr. W. D. Coolidge for supplying the molybdenum tube, and to Dr. C. S. Brainin for the loan of the sheet palladium that was used.

PHŒNIX PHYSICAL LABORATORY,
COLUMBIA UNIVERSITY,
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NEW BOOKS.

The Science and Practice of Photography. By JOHN R. ROEBUCK. New York, D. Appleton and Co., 1918. Pp. xiv + 298.

This book was written for class room use in a course given by the author in the Department of Physics at the University of Wisconsin. The work consists of two parts; the first is a general treatise on photography, while the second part, about one fifth of the book, is a manual of experiments to be performed by the student. The book is planned for students who have had elementary courses in physics and chemistry; no mathematics beyond elementary algebra is needed.

While the book was written primarily as a text for students, many who are interested in photography will find it very instructive reading, especially those who are interested in a more scientific treatment than is usually given in books on this subject. Many references to recent researches are freely given and the results are well summarized. Most of this material, because it was so scattered through the literature of photography, was usually not available to the amateur. This book should find many eager readers.

O. M. S.

Radio Communication, Theory and Methods. By JOHN MILLS. New York, McGraw-Hill Book Co., 1917. Pp. xi + 205.

This book is based on a course of lectures given soon after the United States entered the war, to employees of the Western Electric Company who were being trained for radio work. Since then it has been adopted as a text by the War Department for those men who have had some training in electrical engineering.

Methods used in radio work have been improved so rapidly that the older treatises do not cover modern methods. This book contains much that cannot be found elsewhere except by going through a great many original papers. For example the explanation of the use of the vacuum tube both as a detector and as an amplifier of alternating currents without distortion, is not contained in any of the standard treatises.

The author tries to adapt the book to those who have not had calculus by using the vector operator j and the differential operator p and using only algebraic methods. However, short-cut mathematical processes are of doubtful value to the beginner and certainly one who has had calculus and knows something about the vector methods of solving differential equations as used by engineers will more quickly grasp the processes used. The author has pur-

posely left some of the most important chapters, on the vacuum tube, on detection, on undamped waves, and on radio telegraphy and telephony almost entirely free from mathematical methods.

The author has aimed throughout to "present fundamental principles and methods rather than detailed instruction as to apparatus and its operation." And this aim is fulfilled excellently. In an appendix there is a chapter on transmission over wire circuits and some graded problems for the use of the student.

This interesting book on one of the most interesting branches of electricity will certainly be welcomed by those physicists and and engineers who have not had the time or opportunity to keep in touch with the principles involved in the modern methods.

O. M. S.